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Kumaraswamy Burr Type X Distribution and Its Properties

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Abstract: In this work, we introduce a new model Kumaraswamy Burr-Type X (Kum-BX) from the so-called Kumaraswamy-G family of distributions. Kum-BX serve as an alternative to Kumaraswamy-Weibull model, which is very flexible distribution that has increasing, decreasing and bathtub shapes in the hazard function. Several properties of this new model were provided. The generalization of densities of this new four parameter distribution, the expression for the r^{th} moment, m.g.f, Renyi entropy, and the order statistics were established. A simulation study at different sample sizes with parameter values was done to validate and compare the mean errors where increasing the sample size decreases the error and we also considered the MLE and Bayes methods to estimate the parameters of the new model and its flexibility with an application to real data set was applied to illustrate its usefulness for recommendation in agricultural, medical and engineering areas respectively.

Keywords: Bayesian Analysis, Burr-Type X, Kumaraswamy Burr-Type X, moments, maximum likelihood estimation, simulation study

1. INTRODUCTION

The general concept of this work was aim at developing a new model that encloses the way of its performance towards the betterment of many sub-models which are generally known for a lifetime dataset, which includes the so-called new four-parameter distribution that motivates us with a remarkable features and flexibility based on its properties. Although, for a new class of generalized family of distributions proposed by [1]. The first motivation to introduce this four-parameter model with some strong motivation which are physical and applicable to derive more generalized models and families using the following function of equation:

$$F(t;X,W)=\int_0^{G_1(t;W)} g_2(x;W)dx, 0<x<1, -\infty<t<\infty . \quad (1)$$

Where $G_1(t;W)$ and $g_2(x;W)$ "are the cdf and pdf of the baseline distribution, $G_2(x;X)$ and $g_2(x;W)$ for the generator distribution X, which is the parameter vector of the generator distribution and W is the parameters vector of the baseline distribution [1]". The model proposed by [2], is an old-time probability distribution two random processes with applications to hydrological data. The Kumaraswamy or" Kum" distribution received quite few attention for decades which fits

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hydrological and climatological lifetime datasets [3]. The C.D.F and P.D.F of the Kum distribution, say Kum(ϕ,ψ), is given by:

$$F_{Kum}(t;\phi,\psi) = 1 - (1 - t^{\phi})^{\psi}. \quad (2)$$
$$fKum(t;\phi,\psi) = \phi\psi t(\phi-1)(1 - t\phi)(\psi-1). \quad (3)$$

Where $t\epsilon(0,1)$ We start with the Kum model on the interval (0,1), having P.D.F and C.D.F with two shape parameters $\phi,\psi > 0$ defined by K distribution [3] which fails to accept the fact that beta does not strictly fit hydrological data such as daily rainfall etc. This model is unimodal, increasing, decreasing and constant based on the values of its parameters. It has shown that both the beta and Kumaraswamy distribution have same shape properties [2]. The K model it's not as famous as beta model to statisticians over the years [3]. Also, he emphasizes some advantages of Kum model over the beta model, where he stated that: The normality assumption constant is easy. It has an easy formula for the model. A quantile probability functions does not consist any special functions. A simplicity formula in generating a random numbers and moments of order statistics and L-moments.The rest of the chapter is organized as follows. In section 2, we provide the properties, the expansions for the P.D.F and C.D.F, the moments and moment 0 generating function and deals with order statistics in sections 3, the Re nyi Entropy, Quantile function, Skewness and Kurtosis, in section 4. In section 5, we employed the simulation study, Bayesian and maximum likelihood estimations for comparison of the new model with some sub and existing models applying a real dataset. Finally, concluding remarks in section 6 and acknowledgment are addressed in section 7 respectively.

1.1. Kumaraswamy-G (Kum-G) Family of Distribution

Kum-G family received a quite well attention in the recent years in the area of applied statistics proposed by [4], having C.D.F and P.D.F define respectively as:

$$F_{Kum-G}(t;\phi,\psi) = 1 - [1 - G(t)^{\phi}]^{\psi}. \quad (4)$$
$$fKum-G(t;\phi,\psi) = \phi\psi g(t)G(t)(\phi-1)[1 - G(t)\phi](\psi-1). \quad (5)$$

where, $t > 0,\ g(t) = \frac{d}{dt}G(t)$ and $\phi,\psi > 0$ denotes the shape parameters on the notion for other parameters in the baseline distribution. Kum-G class family by [4], received a quite well attention to researchers in the last few decades in the literatures by some authors, like: [5, 6, 7, 8, 9, 10, 11, 12] respectively.

1.2. Motivations for Choosing Kumaraswamy-G Family.

Furthermore, the basic motivations for the Kum-G family in practice are as follows:
To make the kurtosis more flexible compared to the baseline model;
To produce a skewness for symmetrical distributions;
To construct heavy-tailed distributions for modeling real data;
To generate distributions with symmetric, left-skewed and right-skewed.
To define special models with all types of the hazard rate function;
To provide consistently better fits than other generated models under the same underlying distribution.

1.3. Burr Type X (BX) Distribution

This model was proposed by [13] and revisited by [14], whom contribute a lot to family of continuous distributions and played a vital role in medical, survival and reliability analysis. The second motivation arises to the failure or hazard rate function having some important features including the Burr-Type X (BX) by [13] property which exhibits the fitting of engineering and medical datasets, more especially big data, as it was recently observed by [14, 18], that BX model is very effective and versatile in modeling reliability strength and lifetime datasets. They also thought BX model can be exponentiated Rayleigh (ER) [19] or generalized Rayleigh (GR) [20], but they prefer calling it BX model due to its suitability to Burr family of distribution and properties. They observed that the GR or BX model with two-parameter also has a quite common properties with gamma, generalized exponential and Weibull distributions respectively. Although, the BX model density and cumulative distribution functions have a simple close form and its also has a convenient and flexible feature in modeling censored (incomplete) data, unlike gamma, GE and Weibull distributions. The two-parameter BX has a monotonically increasing and decreasing hazard function features, which can be used for practical aspects in statistical distribution and modeling of applications. Recently, authors have been studying BX model due to its ability and flexibility in modeling reliability datasets such as, [18]. Surless and Paglet [14] introduced the two parameter Burr type X distribution where authors like: [16, 17] made some extension. The Burr type X distribution can be used in modeling general lifetime data. Based on this model distribution in [13], whose CDF and PDF are given by:

$$F_{BX}(t;\vartheta,\tau) = \{1 - e^{-(\tau t)^2}\}^{\vartheta}. \qquad (6)$$

$$f_{BX}(t;\vartheta,\tau) = 2\vartheta\tau^2 t e^{-(\tau t)^2}\{1 - e^{-(\tau t)^2}\}^{(\vartheta-1)}. \qquad (7)$$

where ϑ and τ are the shape and scale parameters respectively. The two parameters BX or GR model in equation 7 above, implies $BX(\vartheta,\tau)$, if $\vartheta = 1$, BX distribution reduces to a well-known one parameter Rayleigh distribution. If $\vartheta \neq 1$, this plays the role of τ, which is the scale parameter. On the other hand, if $\vartheta \leq \frac{1}{2}$, therefore the P.D.F of BX model will be decreasing function of the model while for $\vartheta > \frac{1}{2}$, it proves that it is a unimodal right skewed function. If it has a mode written as $\frac{t_o}{\tau}$, where t_o is called the non-linear systems of equation given as:

$$1 - 2t^2 - e^{-t^2}(1 - 2\vartheta t^2) = 0.$$

The above mode of BX model shows clearly that it is decreasing function of the scale parameter τ and also the increasing function of ϑ respectively. The P.D.F figures at different forms resemble that of Weibull and gamma functions. Moreover, the median of the BX model occurs when the given quantile at:

$$\left[-\frac{1}{\tau}\ln(1-\frac{1}{2})^{\frac{1}{\vartheta}}\right]^{\frac{1}{2}}$$

and this also shows that the non-increasing form of τ, and the non-decreasing form of ϑ.

1.3.1. Motivations for Choosing Burr Type X as a Baseline Distribution

The motivations for choosing Burr Type X distribution are as follows:

It is quite common to the two-parameter gamma, Weibull and generalized exponential distributions.

The density function of Burr Type X has close form.

The Burr Type X can be used very conveniently even for censored data unlike Burr Type XII [41] which yields to fit only uni-modal dataset.

Unlike Burr Type XII, gamma, Weibull and Generalized exponential distributions, it has non-monotone hazard form which can be very useful in many practical applications [41].

The Burr Type X hazard function is monotonically increasing, monotonically decreasing and bathtub shapes unlike Burr Type XII which has similar properties having a non-monotone h(t) [42]and also allocate so many shapes, as well as gamma, Weibull and Generalized exponential distributions with only monotonically increasing, monotonically decreasing shapes respectively.

The Burr Type X model, as a special case to the due to the relative flexibility of its hazard function and the ease for estimation of its parameters, ever since it has been widely used for analyzing reliability and agricultural lifetime data.

1.4. Kumaraswamy Burr Type X (Kum-BX) Distribution

We Introduced a new model called Kum-BX with four parameters ($\theta = \phi,\psi,\vartheta,\tau$), which was a sub-model to the so-called Kumaraswamy-G family stated above by [4]. our proposed new model by the method of adding parameter and model by [23], having a P.D.F given as:

$$2 -(\tau t)^2 -(\tau t)^2 (\vartheta\phi-1) \; f_{Kum-BX}(t,\phi,\psi,\vartheta,\tau) = 2\phi\psi\vartheta\tau \; te \; \{1 - e\} \times \{1 - (1 - e-(\tau t)^2)(\vartheta\phi)\}(\psi-1), \quad (8)$$

A random variable T, with P.D.F ($\theta = \phi,\psi,\vartheta,\tau$) is said to follow the Kum-BX with parameters $\phi,\psi,\vartheta,\tau > 0$ or T ~ Kum − BX(ϕ,ψ,ϑ,τ). The corresponding cumulative distribution function (C.D.F), survival function S(t) or Reliability function R(t), hazard function h(t), cumulative hazard function H(t) are all given as follows:

$$F_{Kum-BX}(t,\phi,\psi,\vartheta,\tau) = 1 - (1 - \{1 - e^{-(\tau t)2}\}^{\vartheta\phi})^{\psi}. \quad (9)$$

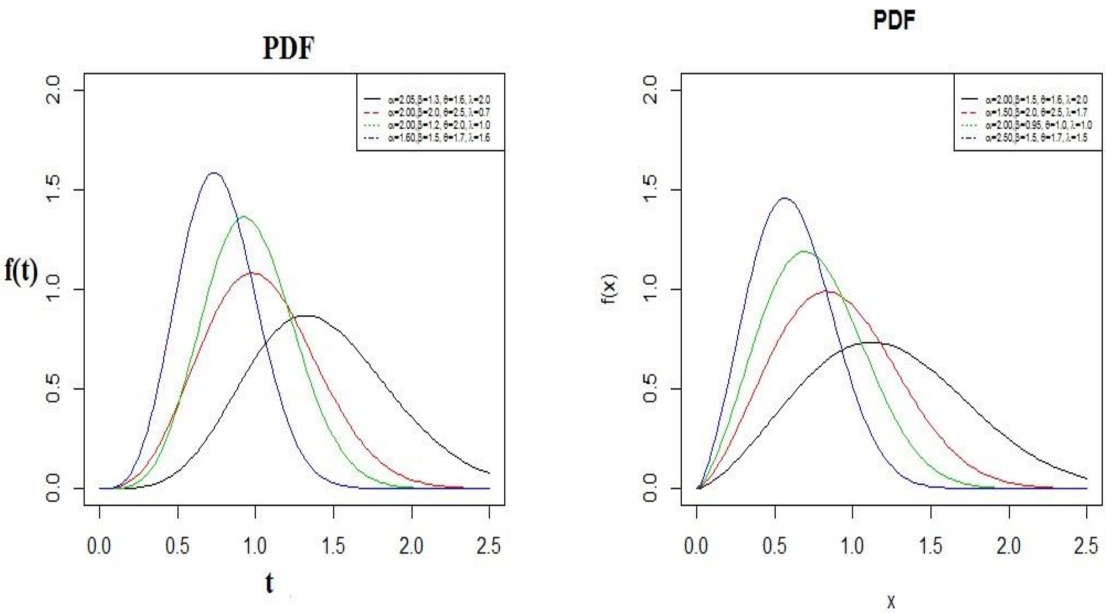
$$S_{Kum-BX}(t,\varphi,\psi,\vartheta,\tau) = 1 - \{1 - e^{-(\tau t)^2}\}^{\vartheta\varphi})^{\psi}.$$
$$R_{Kum-BX}(t,\varphi,\psi,\vartheta,\tau) = S^{Kum-BX}(t,\varphi,\psi,\vartheta,\tau). \quad (10)$$
$$h_{Kum-BX}(t,\varphi,\psi,\vartheta,\tau) = \frac{2\varphi\psi\vartheta\tau^2 te^{-(\tau t)^2}\{1 - e^{-(\tau t)^2}\}^{\varphi-1}}{\{1 - (1 - e^{-(\tau t)^2})^{\vartheta\varphi}\}}.$$
$$H_{Kum-BX}(t,\varphi,\psi,\vartheta,\tau = -\ln[St] = -\ln\left[\{1 - (1 - e^{-(\tau t)^2})^{\vartheta\varphi}\}\right]$$

1.4.1. Motivations for KBX Distribution

The motivation for choosing KBX model are as follows:
Model flexibility due to number of shape parameters provided.
Ability to tackle and solve an old problem by proposing a joint model.
We were motivated to introduced a new model called the KBX model with four ($\theta = \phi,\psi,\vartheta,\tau$) parameters distribution which generalized many of its baseline distributions and sub-models as well as the properties and flexibility of the model.
The fact that, this new model consists of several varieties of model fitting many areas of applied statistics, engineering, medicine and agriculture.
It was noted from the above C.D.F of the new model equation 9, $F_{Kum-BX}(t,\theta)$ $m_1 Y$ as $t \to 0$ also $1 - F_{Kum-BX}(t,\theta) \sim m_2 \exp-(\tau t)^2$ as $t \to \infty$ where m_1 and m_2 denotes the model constraints.These two models serve as a generalization to other existing models which are very flexible and versatile models with properties of their densities which can be expressed as a mixture of many sub-models of Kumaraswamy-G families. For example, Burr Type X, Beta exponential, generalized exponential and Rayleigh distributions respectively. This new model Kum-BX encloses a lower tail performance for the exponentiated Rayleigh model and the upper tail acts like the Burr-Type X model. The new model Kumaraswamy Burr-Type X motivates us with its great varieties and has a common property with Kumaraswamy-Weibull(kum-W) proposed by [18] model relating to flexibility and suitability towards failure data where these two are sub-model to each other but kum-W is better than our proposed new model which Kum-BX serve as an alternative to Kum-W model. We consider the last motivation, is empirical based we provide the model Kum-BX be more profitable than some six (6) among of the known two-parameter and threeparameter models with respect to a real censored data.

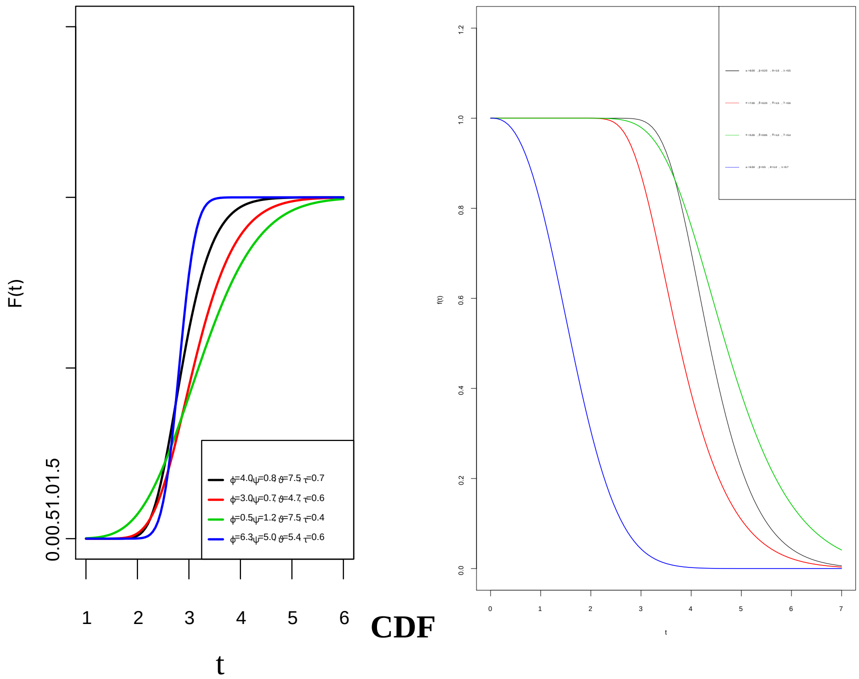
Additionally, KBX is the modification of Beta Kumaraswamy Burr Type X (BKBX) and also its reduce sub-model, where its more advantageous and less time consuming in modeling life time data and simulations based on the comparison with the original one, which is the BKBX model from the generalized family proposed by [37] and also the baseline BX model with two parameters ϑ and τ respectively.

On the other hand, The Figure 8 and Figure 9 below, described the shapes of the PDF and CDF for the given parameter values. These functions represent different kind of forms depending on choosing values of KBX model parameters. We noticed that the additional shape parameters provide a high a level of flexibility. The figures below, shows the failure rate or hazard function of Kum-BX model increasing and decreasing or bathtub shapes. The Kum-BX distribution with four parameters is more flexible than the original Burr type X distribution with four and two parameters due to flexibility of Kumaraswamy distribution with two parameters, that leads more smooth and vital. This new model will be useful in modeling and analyzing real life, censored and uncensored data in
110medical, engineering, pure science and agricultural areas.



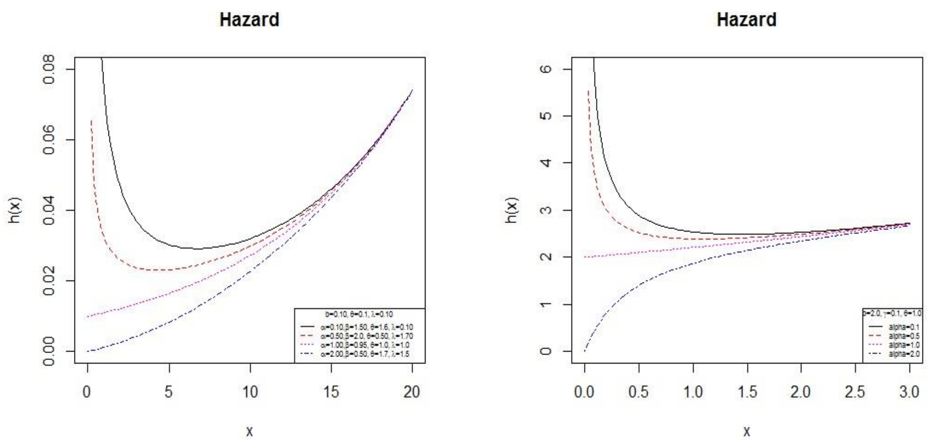
(a) Probability Density Function 1. (b) Probability Density Function 2.
Figure 1: Plot of the Kum-BX probability density function (PDF) for different parameter values of ϕ, ψ, ϑ and τ.

CDF of Kum−BX



(a) Cummulative Distribution Function. (b) Cummulative Distribution Function.

Figure 2: Plot of the Kum-BX cumulative distribution function (CDF) for different parameter values of ϕ,ψ,ϑ and τ.



(a) Hazard Function 1. (b) Hazard Function 2.

Figure 3: Plot of Kum-BX hazard function at different parameter values of ϕ,ψ,ϑ and τ.

2. PROPERTIES

2.1. Limit Behavior

Lemma 2.1. *Refer to equation 8, the limit behavior for the probability density function of the Kum-BX with four parameters when $t \to 0$ and $t \to \infty$*

lim $\theta(t,\phi,\psi,\vartheta,\tau) = 2\phi\psi\vartheta\tau 2te{-}(\tau t)2\{1 - e{-}(\tau t)2\}(\vartheta\phi{-}1)$ $t{\to}\infty{-}(\tau t)^2$ $(\vartheta\phi)$ $(\psi{-}1)\{1 - (1 - e)\} = 0$

$2 - (\tau t)^2$ because lim = $2\phi\psi\vartheta\tau$ te = 0 t→∞

on the other hand, as $t \to \infty$, we observed that by substituting the limit $t \to 0$ with $t \to \infty$, the above expression limit behavior deduces to zero.

2.2. The Expansion oF PDF and CDF

For the Kum-BX parameters:
$\theta = \phi, \psi, \vartheta, \tau$.
We adopt the use of binomial expansion by [19]. If $\psi > 0$ is a real non-integer, for the expansion series representation we use,

$$[1 - F_{Kum-BX}(t)^{\vartheta\varphi}]^{\psi-1} = \sum_{i=0}^{\infty}(-1)^i \binom{\psi-1}{i} (1-(1-\{1-e^{-(\tau t)^2}\}^{\vartheta\varphi})^{\psi})^{1/2} \quad , \quad (11)$$

The binomial expansion coefficient is defined for real number. From the expansion in equation 11 we can write the Kum-BX density function expansion as:

$$fKum - BXexp. = fKum{-}BX \sum_{i=0}^{\infty} w_i F_{BX}(t)^{\vartheta\phi(i+1)-1}, \quad (12)$$

If the parameters ϕ and ψ are integers equation 11 shows that density function of Kum-BX equals to the density of the BX distribution multiply by the infinite power series of the cumulative density function of BX model.

Where the coefficients are : $\omega_i = \omega_i(\varphi\psi) = (-1)^i \varphi\psi \binom{\psi-1}{i}$ and $\sum_{i=0}^{\infty} \omega_i = 1$,

Finally, it written as : $f_{Kum-BX}(t;\phi\psi\vartheta\tau) = 2\phi\psi\vartheta\tau^2 \sum_{l=0}^{i-1} \omega_i [f_{BX}(t,\vartheta\tau)]^l$. (13)

On the other hand if the parameters are non-integer, we can expand:

$FKum{-}BX(t;\phi,\psi,\vartheta,\tau)^{\vartheta\phi(i+1)-1}$, We can expand the C.D.F with the below given form:

$$F_{Kum-BX}(t) = 1 - \{1 - e^{-(\tau t)^2)^{\vartheta\phi}}\}^{\psi}.$$

we also used the incomplete beta function [19] and obtained:

$$I_y(\varphi,\psi) = 2y^{\varphi}\varphi\psi\vartheta\tau^{2t} \sum_{0}^{\infty} \binom{\varphi-1}{\psi} \frac{(-1)y}{(\varphi+1)}.$$

On the other hand, the expansion can be derived as proposed by [19],

$$FKum{-}BX(t) = \sum_{\vartheta=\phi}^{\phi+\psi-1} \sum_{\tau=0}^{\phi+\psi-1-\vartheta} \eta_{\vartheta,\tau}(-1)^{\tau}[[-FKum{-}BX(t)]^{\vartheta+\tau}. \quad (14)$$

$Where,\ \eta_{\vartheta,\tau} = \binom{\varphi+\psi-1}{\vartheta}\binom{\varphi+\psi-1-\vartheta}{\tau}$.

2.3. Some few Special Sub-models of Kum-BX

The Kum-BX has several special cases or sub models but among are few stated below:
When $\phi * \vartheta = \alpha$, $\psi = 2$, therefore, equation 8, above will transforms to Kumaraswamy-Weibull distribution with four parameters.

When $\vartheta = \tau = 1$, therefore, equation 8, above will reduces to Kumaraswamy distribution with two parameters.

When $\phi = \psi = 1$, therefore, equation 8, above will reduces to Burr type X distribution with two parameters.

When $\phi = \psi = 2$, therefore, equation 8, above will reduces to generalized exponential distributions with two parameters.

When $\phi = \vartheta = \tau = 1$, therefore, equation 8, above will reduces to Rayleigh distribution with one parameter.

3. The Probability Weighted r^{th} Moments

Probability weighted r^{th} moments (PWMs), was initially introduced by [13], defined to be the expectation of some functions of a random variable x and y defined. The $(m,n,r)^{th}$ PWM of T is defined by:

$$\Gamma\varphi = \int_{-\infty}^{\infty} t^m F(t)^n [1-F(t)]^r f(t)dt \tag{15}$$

From the above equations 13 and 14 the s^{th} moment of T can be written either
As

$$E(T^s) = \int_{-\infty}^{\infty} t^s f_{K-G}(t;\varphi,\psi)dt$$
$$= \sum_{j=0}^{j+\psi-1} \eta_l \int_{-\infty}^{\infty} t^s [F_{KBX}(t;\theta_2)]^l f_{K-G}(t;\varphi,\psi)dt.$$

$j+\psi-1$
$= X\ \eta l \Gamma s,l,0.$
$j=0$

Were,

$$\eta_l = \binom{\varphi+\psi-1}{\vartheta}\binom{\varphi+\psi-1-\vartheta}{\tau}$$
$$\Gamma_{\varphi,\psi,r} = \int_{-\infty}^{\infty} t^m F_{KBX}(t;\theta_2)]^n f_{K-G}(t;\varphi,\psi)dt$$

is the PWM of Kum-G(ϕ,ψ) distribution. Therefore the moments of the KumBX (ϕ,ψ,ϑ,τ) can be transform by the PWMs of Kum-G(ϕ,ψ). This method can be used for estimating parameters quantiles of generalized distributions.

Proceeding as the above s^{th} moment of the r^{th} order statistic $T_{r:n}$ in the random sample of size n.
$k+l$
$E(Trs{:}n) = X\ \xi z\ \Gamma s,0,z$
$z=0$ Where, ψ_j, η_l, ξ_z are define above.
These follows an incomplete beta function [3]:

$$\Gamma\varphi = \int_0^{\infty} e^{-y} y^{\varphi-1} dy \tag{16}$$
$$E(X^r) = 2\varphi\psi\vartheta\Gamma(r/2+1)\tau^r C_1 \frac{1}{(k+1)^{\frac{r}{2}+1}}$$

ψ $\vartheta\phi$

Where, $C_1 = \sum_{i=0}^{\varphi-1} \sum_{j=0}^{\varphi(i+\psi)-j} \sum_{k=0}^{(j+1)} (-1)^{i+j+k} \binom{\varphi-1}{i} \binom{\varphi(i+\psi)-j}{j}$

3.1**. Moment Generating Function (MGF)**
MGF of Kum-BX distribution can be obtained and expressed in form of exponential Kum-G family of distribution from the results we obtained in the moments above using equation (14) above,

(17)

$$Where,\ e^t x = \sum_{s=0}^{\infty} \frac{t^s x^s}{s!}$$

$_s$t and $E(X^s) = \sum_{s=0}^{\infty}$ s
s!

$$M_T(s) = \sum_{s=0}^{\infty} \frac{t^s x^s}{s!} \int_0^{\infty} e^t x f(t) dt$$

$$M_T(s) = \sum_{s=0}^{\infty} \frac{2\varphi\psi\vartheta\Gamma(s/2+1)t^s}{\vartheta\tau^s s!} C_1 \frac{1}{(k+1)^{\frac{s}{2}+1}}$$

(18)

$$C_1 = \sum_{i=0}^{\varphi-1} \sum_{j=0}^{\varphi(i+\psi)-j} \sum_{k=0}^{\vartheta\varphi(j+1)} (-1)^{i+j+k} \binom{\varphi-1}{i} \binom{\varphi(i+\psi)-j}{j} \quad (19)$$

Where $M_T(s)$ is the MGF of a KBX distribution. Where $M_T(s)$ is the MGF of a Kum-BX distribution.

3.2. Order Statistics
Some authors like: [14, 16] adopted this procedure by assuming a random sample from a population $T_1, T_2, ., T_n$ from Kum-BX distribution with four parameters $\theta = \phi, \psi, \vartheta, \tau$. We denote $T_{r:n}$ as the r^{th} order statistics, given the P.D.F as:

$$f_{r:n} Kum-BX(t) = \sum_{y=0}^{k+j} \xi_y^0 f^{Kum-BX}(t;(y+1)). \quad (20)$$

Therefore,

$$\xi'_y = \sum_{j=0}^{k+q-1} \sum_{k=0}^{\infty} \frac{(-1)^{y+1}\gamma_{j,k}}{y+1} \binom{k+j}{y},\ \xi_y = \xi'_y(y+1)$$

$$\gamma_{j,k} = \frac{n!}{(r-1)!(n-r)!}, \sum_{k=0}^{q-r} (-1)^k \binom{q-r}{k} \tau_j d_{j+r-1,k}.$$

Where τ_j and $d_{j+r-1,k}$ were stated and defined in the equations above.

4. Renyi Entropy
An entropy for a given random variable is called the measure of uncertainty used in many areas of research like applied statistics and engineering [2]. It is given by definition:

$$I_R(\epsilon) = (1-\epsilon)^{-1} \log \left(\int_{-\infty}^{\infty} f(t)^{\epsilon} dt \right). \quad (21)$$

where > 0 and ϵ /= 1 for more details refer to [14]. Using binomial expansion in 8 we can express it by denoting:

$$k_i = e^{-(\tau t)^2}, \quad M_i = (1-e^{-(\tau t)^2})^{(\varphi-1)}, \quad R_i = (1-e^{-(\tau t)^2})^{(\vartheta\varphi)}.$$

$$\begin{aligned}\int_0^\infty f(t)^\gamma dt =& \big(2)^\gamma\big)\int_0^\infty \Big[\varphi\,\psi\,\vartheta\,\tau^2 t_i e^{-(\tau t_i)^2} M_i^{(\vartheta\varphi-1)}[1-R_i]^{\psi\kappa-1}\Big]^\gamma dt.\\ =&\big(2\varphi\,\psi\,\vartheta\,\tau^2\big)^\gamma \sum_{i=0}^{(\nu-1)\gamma}\binom{\gamma(\psi-1)}{i}(-1)^i\int_0^\infty t_i^\gamma e^{-\gamma(\tau t_i)^2}\\ &\times\Big[M_i^{(\vartheta\varphi-1)}\Big]^\gamma\Big[1-R_i\Big]^\gamma dt.\\ =&\big(2\varphi\,\psi\,\vartheta\,\tau^2\big)^\gamma \sum_{i=0}^{(\nu-1)\gamma}\binom{\gamma(\psi-1)}{i}(-1)^i\int_0^\infty t_i^\gamma e^{-\gamma(\tau t_i)^2}\\ &\times\Big[M_i^{(\vartheta\varphi-1)}\Big]^\gamma\Big[1-(1-(M_i)^{(\vartheta\varphi)})\Big]^\gamma dt.\\ =&\big(2\varphi\,\psi\,\vartheta\,\tau^2\big)^\gamma \sum_{i,j=0}^{(\psi-1)\gamma}\binom{\gamma(\psi-1)}{j}\binom{\gamma(\vartheta\varphi-1)i}{i}(-1)^{i+j}\\ &\times\int_0^\infty t_i^\gamma e^{-\gamma(\tau t_i)^2}dt.\end{aligned} \tag{22}$$

Therefore, we obtained the *Renyi´* entropy by substituting Equation (22), in Equation (**11**) as follows:

$$\begin{aligned}I_R(\gamma) =&\big(1-\gamma\big)^{-1}\ln\Big\{\big(2\varphi\,\psi\,\vartheta\,\tau^2\big)\\ &\times\sum_{i,j=0}^{(\psi-1)\gamma}\binom{\gamma(\psi-1)}{j}\binom{\gamma(\vartheta\varphi-1)i}{i}(-1)^{i+j}\\ &\times\int_0^\infty t_i^\gamma e^{-\gamma(\tau t_i)^2}dt\Big\}.\end{aligned}$$

Thus, the R´enyi entropy is, (23)

$$\begin{aligned}I_R(\gamma) =&\frac{\gamma}{1-\gamma}\ln\Big(2\varphi\,\psi\,\vartheta\Big)+\ln(2\tau)\\ &+\frac{1}{1-\gamma}\ln\Big\{\sum_{i,j=0}^{(\psi-1)\gamma}\binom{\gamma(\psi-1)}{j}\binom{\gamma(\vartheta\varphi-1)i}{i}(-1)^{i+j}\\ &\times\int_0^\infty t_i^\gamma e^{-\gamma(\tau t_i)^2}dt\Big\}.\end{aligned} \tag{24}$$

4.1. Quantile Function

Let $Q_{\phi,\psi,\vartheta,\phi}(u)$ be the Kumaraswamy Burr-Type X quantile function with parameters ϑ and ϕ and let u (0,1) be the parameter interval. The quantile function is the inverse of C.D.F of the new model given as :

$$\begin{aligned}1-\Big\{1-\Big(1-e^{-(\tau t)^2}\Big)^{\vartheta\varphi}\Big\}^\psi =&u,\\ 1-\Big(1-e^{-(\tau t)^2}\Big)^{\vartheta\varphi} =&\Big(1-u\Big)^{1/\psi},\\ \Big\{1-e^{-(\tau t)^2}\Big\}^{\vartheta\varphi} =&1-\Big(1-u\Big)^{1/\psi},\\ e^{-(\tau t)^2} =&\Big[1-\Big(1-u\Big)^{1/\psi}\Big]^{1/\vartheta\varphi},\\ -(\tau t)^2 =&\ln\Big\{1-\Big[1-\Big(1-u\Big)^{1/\psi}\Big]^{1/\vartheta\varphi}\Big\}.\end{aligned}$$

Therefore, the quantile function of Kum-BX distribution is given below by taking the inverse making" t" the subject of the formula.

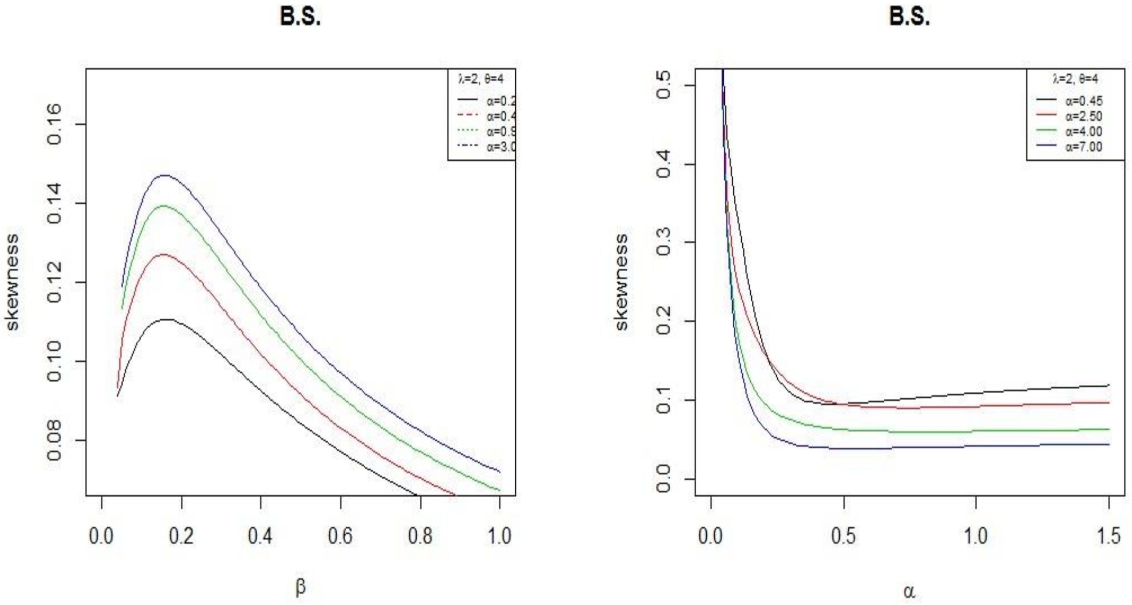
$$t=\frac{1}{\tau}\times\Big[-\ln\Big\{1-\Big[1-\Big(1-u\Big)^{1/\psi}\Big]^{1/\vartheta\varphi}\Big\}\Big]^{1/2}. \tag{25}$$

$Where,\ Q_{\varphi,\psi,\vartheta,\varphi}(u) = 1-I_u^{-1} = \big(Q_{(u,\varphi\psi)}\big)^{1/\psi}$.

We also denote Z to be the random variable for Kumaraswamy. The above quantity is used to generate Kum-BX random variates due to the existence of the Kumaraswamy generators family to obtain Kumaraswamy random variables. On the other hand, If Z is a random variable follows a Kumaraswamy distribution with parameters, then Z is called the Kumaraswamy random variable with ϕ and ψ parameters, therefore,

$$T = [[1 - Z_{\varphi,\psi}]^{1/\vartheta\varphi}]^{1/2}.$$

This follows a the Kum-BX random variates model and also, equation 25 above, we can generalized and conclude that the median of Kum-BX is m of X is $m = Qu(1/2)$.



(a) Skewness Shape 1. (b) Skewness Shape 2.

Figure 4: Plot of the Kum-BX Skewness at different parameter values ϕ,ψ,ϑ and τ.

4.2. **Skewness and Kurtosis**

The Bowley skewness [20], is actually a statistical procedure to find the positive or negative skewed distribution on based on your data. It is the most popular tool procedure in finding a skewness fit given by.

$$S_k = \frac{Qu(3/4) + Qu(1/4) - 2Qu(1/2)}{Qu(3/4) - Qu(1/4)} \tag{26}$$

On the other hand, we have, the Moors kurtosis [21], can be calculated by using the formula given below:

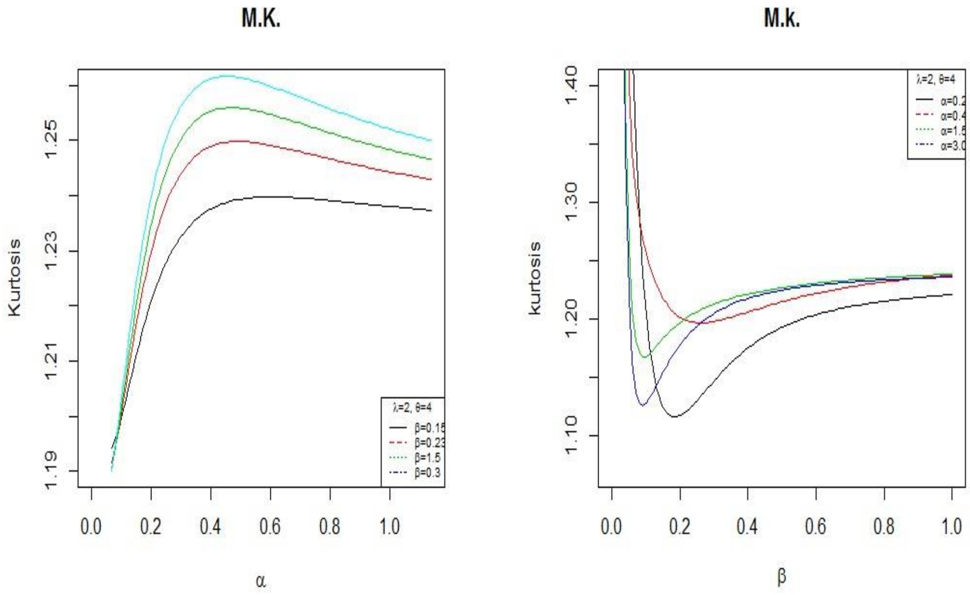
$$M_k = \frac{Qu(3/8) - Qu(1/8) + Qu(7/8) - Qu(5/8)}{Qu(6/4) - Qu(2/8)} \tag{27}$$

4.3. Maximum Likelihood Function (MLE)

Let $T_1, T_2, ..., T_n$ be a random sample of size n with observed values $t_1, t_2, ..., t_n$. We can estimate the Kum-BX model with four parameters $\theta = (\phi, \psi, \vartheta, \tau)$ by the method of maximum likelihood, given the likelihood function as:

$$L(\theta) = \prod_{i=1}^{n} 2\phi\psi\vartheta\tau^2 t_i e^{-(\tau t_i)^2} \prod_{i=1}^{n} \{1 - e^{-(\tau t_i)^2}\}^{(\vartheta\phi - 1)} \times \prod_{i=1}^{n} \left\{1 - (1 - e^{-(\tau t_i)^2})^{(\vartheta\phi)}\right\}^{(\psi - 1)}. \quad (28)$$

for $\phi > 0, \psi > 0, \vartheta > 0, \tau > 0$. Let $t = (t_1, t_2, ,,,, t_r)^T$ be a random sample of size r from Kum-BX with parameters $\Phi = (\phi, \psi, \vartheta, \tau)^T$. Then the log-likelihood



(a) Kurtosis Shape 1. (b) Kurtosis Shape 2.

Figure 5: Plot of the Kum-BX Kurtosis at different parameter values ϕ, ψ, ϑ and τ. function for is given by:

$$\ell(\theta) = n \ln\left(2\varphi\,\psi\,\vartheta\right) + 2n \sum_{i=1}^{n} \ln\left(\tau t_i\right) - \sum_{i=1}^{n} \left(\tau t_i\right)^2 + \left(\vartheta\varphi - 1\right) \sum_{i=1}^{n} \ln\left[1 - e^{-(\tau t_i)^2}\right] + \left(\psi - 1\right) \sum_{i=1}^{n} \ln\left[1 - \left(1 - e^{-(\tau t_i)^2}\right)^{\vartheta\varphi}\right]. \quad (29)$$

By applying the partial derivatives of the equation 29 above with respect to $\phi, \psi, \vartheta, \tau$ components of the score vectors $U_\Phi = U_\phi, U_\psi, U_\vartheta, U_\tau)$ and denoting:

$K_i = \tau t_i$

$M_i = e^{-K_i^2}$ $R_i = 1 - M_i$

equation 29 above deduce to:

$$\ell(\theta) = n\Big\{\ln(2) + \ln(\varphi) + \ln(\psi) + \ln(\vartheta) + 2\ln(\tau)\Big\} + 2n\ln(\tau) + \sum_{i=1}^{n}\ln(t_i) - \sum_{i=1}^{n}(K_i)^2 + \Big(\vartheta\varphi - 1\Big)\sum_{i=1}^{n}\ln\Big(R_i\Big) + \Big(\psi - 1\Big)\sum_{i=1}^{n}\ln\Big[\Big(R_i\Big)^{\vartheta\varphi}\Big]. \quad (30)$$

We obtained the following partial derivatives for each parameter:

(31)

$$U_\varphi = \frac{\partial l}{\partial \varphi} = \frac{n}{\varphi} + \vartheta\sum_{i=1}^{n}\ln\Big(R_i\Big) - \vartheta\Big(\psi - 1\Big)\sum_{i=1}^{n}\frac{\Big(R_i\Big)^{\vartheta\varphi}\ln\Big(R_i\Big)}{1 - \Big(R_i\Big)^{\vartheta\varphi}} = 0 \quad (33)$$

. (32)

(34)

(35)

$$U_\psi = \frac{\partial l}{\partial \psi} = \frac{n}{\psi} + \sum_{i=1}^{n}\ln\Big[1 - \Big(R_i\Big)^{\vartheta\varphi}\Big] = 0.$$

$$U_\vartheta = \frac{\partial l}{\partial \vartheta} = \frac{n}{\vartheta} + \vartheta\sum_{i=1}^{n}\ln\Big(R_i\Big) - \varphi\Big(\psi - 1\Big)\sum_{i=1}^{n}\frac{\Big(R_i\Big)^{\vartheta\varphi}\ln\Big(R_i\Big)}{1 - \Big(R_i\Big)^{\vartheta\varphi}} = 0.$$

$$U_\tau = \frac{\partial l}{\partial \tau} = \frac{2n}{\tau} + \frac{2}{\tau}\sum_{i=1}^{n}(K_i)^2 + \frac{\Big(\vartheta\varphi - 1\Big)}{\tau}\sum_{i=1}^{n}\frac{M_i K_i^2}{R_i} - \frac{2\vartheta\varphi\Big(\psi - 1\Big)}{\tau}\sum_{i=1}^{n}\frac{K_i^2\, M_i\Big(R_i\Big)^{\vartheta\varphi - 1}}{1 - \Big(R_i\Big)^{\vartheta\varphi}} = 0.$$

Where $\omega(.)$ represents the digamma function obtaining the estimations of the score functions in equation 32, 33 34, 35, using the software" MAPLE" to simplify the solution respectively. The provision of interval estimation for the hypothesis test based on the new model unknown parameters, an information matrix of all the Kum-BX distribution.

5. Simulation Study

We adopt the Monte Carlo simulation study to access the performance of the MLE's of (ϕ,ψ,ϑ *and* τ). We generate different n sample observation from the quantile function in equation 25 above of the new model Kum-BX. The parameters are estimated by maximum likelihood method. We considered different sample size n =50, 150, 300 and 500 and the number of repetitions is 5000. The true parameters value as ϕ, ψ, ϑ and τ with three different sets of values, in Table 1, 2 and 3, of below shows the bias and mean squared error (MSE) of the estimate parameters at different parameter values. We observed that, when we increase sample sizes" n" the bias and mean square error for the Kum-BX model given below as:
$(\phi,\psi,\vartheta,\tau)$

decreases with respect to the best estimation.

5.1. **Inverse CDF method**

In this part we employ an algorithm that generate a sample of size n randomly from the $ET - E(ET - E(\phi;\psi;\vartheta;\tau)$ model for the parameters and sample size n. The simulation process has the below steps:

1. Step 1. We set sample size" n", and ($\theta = \phi;\psi;\vartheta;\tau$).
2. Step 2. We set the initial value t_o for a random procedure.
3. Step 3. We set $j = 1$
4. Step 4. We generate $U \sim Uniform\ (0,1)$.
5. Step 5. We set an Update t_o by using the Bayesian Simulation process.
6. Step 6. If $—t_o - t| \leq \epsilon$, (moderately small, > 0 tolerance limit). Then, $t*$ to the desired sample from $F(t)$.
7. Step 7. If $— t_o - t| >$, then, set $t_o = t*$ and then proceed to step 5.
8. Step 8. We repeat procedure steps from 4-7, for $j = 1, 2,...,n$ and obtained $t_1,t_2,...,tn$.

are close to zero also the mean square error becomes very much smaller. We simulate the model parameters by given true values as we can see the higher you increase the sample size" n" bias the smaller Mean square error becomes towards zero.

5.2. Simulation Study Results

Likewise, the red colour numbers indicates how the Bias and Root mean square errors are decreasing with increase of the sample sizes(n=50,150,300,500).

Simulation Study Results:

Table 1. The Bias and RMSE on Monte Carlo simulation for Parameters values θ=4,2,5.5,1.5.

Sample size "n"	True parameter value	Mean	Bias	MSE
	α=4	4.2050	0.2050	0.7392
	β=2	2.1534	0.1534	0.6813
	ϑ=5.5	5.6288	0.1288	0.6672
50	λ=1.5	1.5112	0.0112	0.0079
	α=4	4.0637	0.0637	0.2610
	β=2	2.0434	0.0434	0.1596
	ϑ=5.5	5.5981	0.0981	0.3249
150	λ=1.5	1.5063	0.0063	0.0028
	α=4	4.0267	0.0267	0.1183
	β=2	2.0075	0.0075	0.0494
	ϑ=5.5	5.6080	0.1080	0.1112
300	λ=1.5	1.5058	0.0058	0.0010
	α=4	3.9874	-0.0126	0.0387
	β=2	2.0088	0.0088	0.0218
	ϑ=5.5	5.5631	0.0631	0.0534
500	λ=1.5	1.5017	0.0017	0.0004

Likewise, the red color numbers indicate how the Bias and Root mean square errors are decreasing with increase of the sample sizes(n=50,150,300,500).

Table 2. The Bias and MSE on Monte Carlo simulation for Parameters values θ=4,2.5,5.5,2.5.

Sample size "n"	True parameter value	Mean	Bias	MSE
	α=4	4.2529	0.2529	0.8976
	β=2.5	2.7822	0.2822	1.1090
	ϑ=5.5	5.6170	0.1170	0.7653
50	λ=2.5	2.5079	0.0079	0.0214
	α=4	4.0706	0.0706	0.3259
	β=2.5	2.5861	0.0861	0.3362
	ϑ=5.5	5.6071	0.1071	0.3418
150	λ=2.5	2.5085	0.0085	0.0108
	α=4	4.0066	0.0066	0.0964
	β=2.5	2.5257	0.0257	0.1156
	ϑ=5.5	5.5979	0.0979	0.1655
300	λ=2.5	2.5051	0.0051	0.0035
	α=4	3.9995	-0.0005	0.0468
	β=2.5	2.5069	0.0069	0.0493
	ϑ=5.5	5.5931	0.0931	0.1099
500	λ=2.5	2.5072	0.0072	0.0018

Bayesian Estimation Using Gibbs Sampling Approach.

We wish to obtain a sample from the multivariate distribution (1,...,d). We shall call this distribution the target distribution. In Bayesian statistics, the target distribution is the joint posterior distribution. The Gibbs sampler obtains a sample from (1,...,d) by successively and repeatedly simulating from the conditional distributions of each component given the other components. Under conditional conjugacy, this simulation step is usually straightforward [23,26]. The algorithm is as follows:

Table 3. The Bias and MSE on Monte Carlo simulation for the parameter values θ=4,2,5.5,2.5.

Sample size "n"	True parameter value	Mean	Bias	MSE
	α=4 β=2	4.2496	0.2496	0.8488
	ϑ=5.5	2.2251	0.2251	0.8002
	λ=2.5	5.6189	0.1189	0.7252
50		2.5121	0.0121	0.0239
		4.0477	0.0477	0.2524
	α=4 β=2	2.0670	0.0670	0.2330
	ϑ=5.5	5.5858	0.0858	0.3488
150	λ=2.5	2.5039	0.0039	0.0084
		4.0245	0.0245	0.0906
	α=4 β=2	2.0080	0.0080	0.0582
	ϑ=5.5	5.6049	0.1049	0.1411
300	λ=2.5	2.5101	0.0101	0.0031

		4.0017	0.0017	0.0561
	α=4 β=2	2.0097	0.0097	0.0208
	ϑ=5.5	5.5706	0.0706	0.0768
500	λ=2.5	2.5034	0.0034	0.0012

- Initialize with θ = (θ1(0),...,θd(0)).
- Simulate θ1(1) from the conditional distribution π(θ1|θ2(0),θ3(0),...,θd(0)).
- Simulate θ2(1) from the conditional distribution π(θ2|θ1(1),θ3(0),...,θd(0)).
- Continue sampling....
- Simulate θd(1) from the conditional distribution π(θd|θ1(1),θ2(1),...,θd(1)−1).
- Iterate this procedure.

Under mild regularity conditions, convergence of the Markov chain to the stationary distributionπ(θ1,θ2,...,θd). is guaranteed, so after a burn–in period (that is, a number of iterations for which the draws are discarded), the subsequent draws θ = (θ1(1),,θd(1),...,θ1(J),...,θd(J)). can be regarded as realizations from this distribution. As stressed, this procedure is valid in any situation where the requirement is a sample from a multivariate distribution. Applications have been concentrated in Bayesian statistics because the technique gives a sample-based approach to posterior inference in situations where most other approaches are either difficult or impossible [26,27].

According to the Bayes rule, the posterior distribution is proportional to the product of the prior distribution π(θ) and the likelihood L(θ) assuming the mixture model. Thus, the posterior summaries of interest were obtained from the simulated samples for the joint posterior distribution using standard MCMC procedures [25,26,27].

Under mild regularity conditions, convergence of the Markov chain to the stationary distribution$\pi(\theta_1,\theta_2,...,\theta_d)$. is guaranteed, so after a burnin period (that is, a number of iterations for which the draws are discarded), the subsequent draws$\theta = (\theta_1^{(1)},...,\theta_d^{(1)},...,\theta_1^{(J)},...,\theta_d^{(J)})$. can be regarded as realizations from this distribution. As stressed, this procedure is valid in any situation where the requirement is a sample from a multivariate distribution. Applications have been concentrated in Bayesian statistics because the technique gives a sample-based approach to posterior inference in situations where most other approaches are either difficult or impossible [22, 23, 24]. According to the Bayes rule, the posterior distribution is proportional to the product of the prior distribution $\pi(\theta)$ and the likelihood $L(\theta)$ assuming the mixture model. Thus, the posterior summaries of interest were obtained from the simulated samples for the joint posterior distribution using standard MCMC procedures [22].

5.4. Jeffreys Prior

We assume the posterior density function of a gamma distribution with shape and scale parameters" ϕ" and "ψ". Jeffreys prior can be improper for many models. This prior gives strange results for multivariate θ and modifications have been proposed [22, 23].Its violates the likelihood principle and also gives an automated method for finding a non-informative prior for any parametric model, $P(t|\theta)$ [22]. The likelihood function gamma model is given by:

$$L(\varphi,\tau|t) = \frac{\tau^{n\varphi}}{[\Gamma\varphi]^n}\left(\prod_{i=1}^{n} t_i^{\varphi-1}\right)\exp\left(-\tau\sum_{i=1}^{n} t_i\right). \quad (36)$$

for $\phi,\tau > 0$.

We consider the first order derivative of both the parameters $\frac{\partial}{\partial\phi}\log L_i$ and $\frac{\partial}{\partial\tau}\log L_i$ all equating to zero with some fisher information matrix and we obtained the likelihood equations given below:

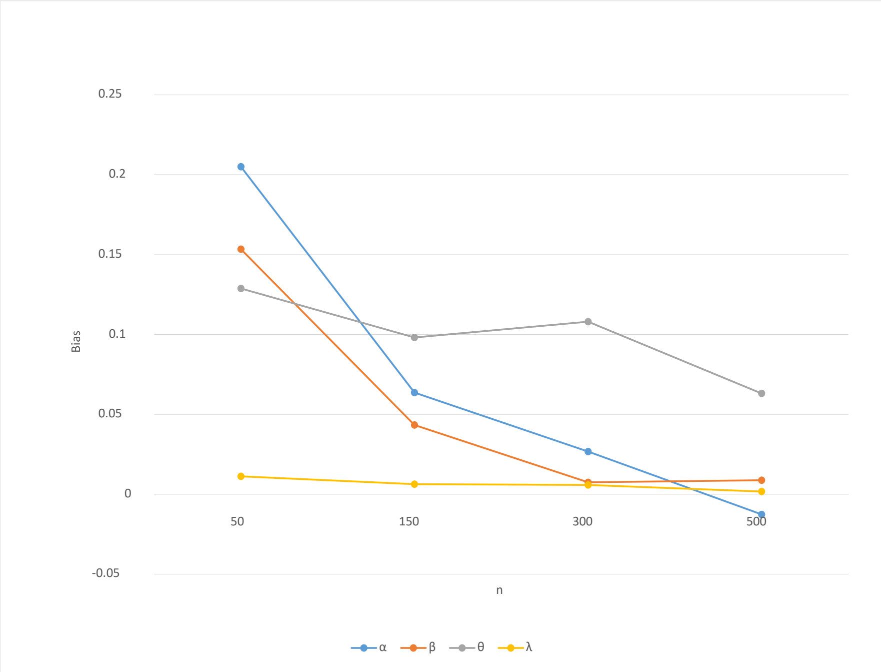


Figure 6: Bias simulated samples of parameter estimates for Kum-BX at different sample sizes

$$\hat{\tau} = \frac{\hat{\varphi}}{\bar{T}} \; and \; \log\hat{\varphi} - \phi(\hat{\varphi}) = \log\left(\frac{\bar{T}}{\tilde{T}}\right). \qquad (37)$$

Where $\phi(j) = \frac{\partial}{\partial j}\Gamma j = \frac{\Gamma j}{\Gamma j}$. The solutions for these equations provide the MLE for the parameters of the Gamma distribution. As closed form solution is not possible to evaluate the numerical techniques [24]. For all cases considered in this paper, we generated 25,000 samples for each parameter of interest. assume a burn in sample of size 10,000 to minimize the effect of the initial values used in the simulation process. The posterior summaries of interest was based on the 10,000 samples, taking every 100th sample to have approximately uncorrelated values [23].

5.5. Results Remarks

Its was shown in table 4, above that the estimates of the parameters were obtained as the median of Gibbs samples drawn from the joint posterior dis- tribution and we used the median rather than the mean since some simulated distribution were skewed in nature. Also we noted that all p values from (HW) Heidelberger and Welch convergence diagnostics [23], criteria do not reject the null hypothesis of stationary of the chains, since they are all larger or equal than

Table 3. The posterior summaries for the parameters of the Kum-BX and other Existing models not including the long-term survival models based on (ACTG) Study [24]:

Model	Parameter	Posterior median	95%*HPD*[a]	DIC	*HW*[c] p value	Geweke's p value
	α	1.3148	(1.4324,2.2399)	144	0.654	0.235 0.251
	β	0.2346	(0.0436,0.2231)		0.352	0.277
Kum-BX	ϑ	2.2565	(1.6425,1.2342)		0.389	0.432

	λ	0.0285	(1.3376,2.3074)		0.089	
	α	3.8116	(1.3456,5.1237)	140	0.314	0.112 0.251
	β	0.0806	(1.2096,1.2432)		0.267	0.307
Beta Weibull	γ	0.9084	(0.4321,0.0987)		0.453	
	λ	2.3126	(1.1214,2.2345)		0.546	0.223
Generalized	α	2.4399	(1,9664,5.2017)	135	0.783	0.384
Exponential	λ	2.3278	(1.2125,3.1348)		0.659	0.384

Results Remarks:
It was shown in table 3, above that the estimates of the parameters were obtained as the median of Gibbs samples drawn from the joint posterior distribution and we used the median rather than the mean since some simulated distributions were skewed in nature. Also, we noted that all p values from (HW) Heidelberger and Welch convergence diagnostics [26], criteria do not reject the null hypothesis of stationary of the chains, since they are all larger or equal than 0.05 level of significance. While the Geweke's p values also suggest convergence, on the other hand these results show that, among all the models considered Generalized exponential distribution has the DIC, Deviance Information Criterion while Kum-BX and Beta Weibull distributions have similar DIC values, where strong evidence shows that Kum-BX has the highest probability density interval (HPDI) [27], and conclude that these models are better fitted by the data [24].

Maximum Likelihood Estimation
Here, we apply data set to clarify the fitness of Kum-BX distribution is a better model than Burr type X and Rayleigh distributions. "The dataset consists of Data from BP Research, [13] and also climatological Illinois dataset by [7] were used for the validation and comparison of the new model with the existing models. The criterion like: Log-likelihood, Akaike Information Criterion, Consistent Akaike Information Criterion and Bayesian information criterion for the data set above so as to compare the models and to check which have least or smaller LL, AIC, AICC and BIC values. The distribution of the data is skewed to the right skewed. From all indication looking at the graph of the comparison below it clearly show that based on this data [13], fits beta and Kumaraswamy family of distributions, on the other hand Kumaraswamy-Burr Type X has the smallest (LL, AIC, AICC and BIC) values among the Exponential, generalized exponential, Beta exponential, Gompertz, Generalized Gompertz models. This suggest that the Kum-BX distribution is very good in modeling right skewed data.

Table 5, above shows MLEs for the individual fitted model for the given data and the estimated (LL, AIC, CAIC and BIC) values and this model as fitted based on the above dataset was left skewed (Skewness= 0.96 and Kurtosis = 1.02). This proves that Kum-BX distribution is a good example for modeling right skewed datasets. Also, the likelihood ratio test with the hypothesis Ho : a = b = 1 versus H1 : a , 1b , 1. Based on the data above $\omega = 18.123 > 5.991 = \chi2;0.05$, we therefore reject the Null hypothesis. For interval estimation and test hypotheses on the parameter, we obtain the observed information matrix $4 * 4$ where, also the results obtain indicates clearly that the Kumaraswamy-Burr Type X is a stronger with highest peak, flexible and vital to its sub-models (Burr Type X and Rayleigh) used here for fitting the data set. On the other hand, we have the 95 percent C.I for the parameters which are: Table 6, above shows the estimated (LL, AIC, CAIC and BIC) values and also the results obtain indicates clearly that the Kumaraswamy-Burr Type X is also stronger based on the histogram with highest peak compare to the other models used here for fitting the data set. On the other hand, we have the 95 percent C.I for the parameters which are:

[0.2024,1.0208],[0.0667,3.0891],[-0.0232,0.2982],[0.6271,2.1234].

This model as fitted based on the above dataset was right skewed (Skewness= 0.98 and Kurtosis = 1.06). This proves this model is a good example for modeling right skewed datasets. Also, the likelihood ratio test with the hypothesis Ho: a = b = 1 versus H1 : a , 1,b , 1. Based on the data above $\omega = 23.025 > 5.991 = \chi2;0.05$, we therefore reject the Null hypothesis.
0.05 level of significance. While the Geweke's p values also suggest convergence [25], on the other hand these results shows that, among all the models considered generalized exponential distribution has the Deviance Information criterion (DIC) while Kum-BX and beta-Weibull distributions have similar DIC values, where a strong evidence shows that Kum-BX has the highest probability density interval (HPDI) [26] and conclude that these models are better fitted by 225 the data [34].

5.6. Maximum Likelihood Estimation

We fit three datasets with these new model KBX to illustrate the potential of the distribution. The real data sets which we used are left-skewed, right skewed and approximately symmetric. In this application we use the method of maximum likelihood to estimate the parameters of the distributions. We use several criteria to compare the three models with the baseline model that is the BX distribution respectively.
In real-life the methodology of any kind of is validate with application to prove the effectiveness and flexibility of the methods, model and approach in the field of applied mathematics, physics and statistics etc. Here, we apply data set to clarify the fitness of KBX distribution by comparing these new models their respective sub-model BX and other existing family of distributions respectively.

The model selection criterion used for comparing the new models with sub models and other existing ones in this chapter are: the -2log-likelihood, Akaike information criterion, consistent Akaike information criterion Bayesian information criterion and Kolmogorov-Smirnov non-parametric goodness-of-fit test statistic for the datasets above so as to check the suitability of the data to the proposed and existing models, which have least value of the criterions, with respect to the right skewed, left skewed and approximate symmetry (normally) 245 respectively.

5.7. Life of Fatigue Fracture of Kevlar 49/Epoxy (Right Skewed Dataset)

The first data, was introduced by [38] were used for the validation and comparison of the new model with the existing models. An investigation of the lifetimes of Kevlar 49/epoxy round weight vessels that are subjected to a steady supported weight until vessel is pressure, usually known as static weakness or stress-crack respectively.

Table 5: The ML estimates, -2log-likelihood, AIC, AICc, BIC and (KSM) for the life of fatigue fracture of Kevlar dataset.

Model	MLE	$-2LL(\theta)$	AIC	AICc	BIC	K-SM	p-value
KBX	ϕ =1.252 ψ =0.886 ϑ= 4.646 τ =3.224	121	252	255	266	0.119	0.418
BKBX	v =1.378 κ = 2.467 ϕ =6.349 ψ= 0.845 ϑ =1.905 τ= 3.135	124	257	258	271	0.139	0.122
BX	ϑ=3.532 τ =1.027	127	265	276	288	0.155	0.146

Table (5), presents the results of lifetimes of Kevlar 49/epoxy round weight vessels dataset and also provides the MLEs of parameters and goodness of fit statistics for BKBX, KBX and BX distributions. From Table (5), the values, $-2LL(\theta)$, (K-SM), AIC, AICc and BIC of KBX is the smallest compared to BKBX and BX distributions, and this means that KBX is the best fit to" life of fatigue fracture of Kevlar 49/Epoxy" dataset and also with the highest $p$value based on the goodness-of-fit test of Kolmogorov-Smirnov non-parametric approach. From Figure (7) it is apparent that the graph of pdf of KBX distribution is most suitable for" life of fatigue fracture of Kevlar 49/Epoxy" dataset. Also, from Figure (8) we can see that the CDF of the KBX distribution is very close to the empirical CDF. Also, we suggested that KBX handles the" life of fatigue fracture of Kevlar 49/Epoxy" data well and it is a good fit for right-skewed

Right Skewed Fitted Models

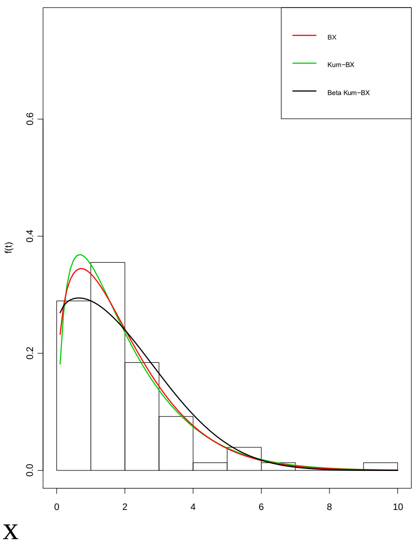


Figure 7: The histogram for Life of Fatigue for Kevlar Right Skewed Dataset.

Empirical Distribution

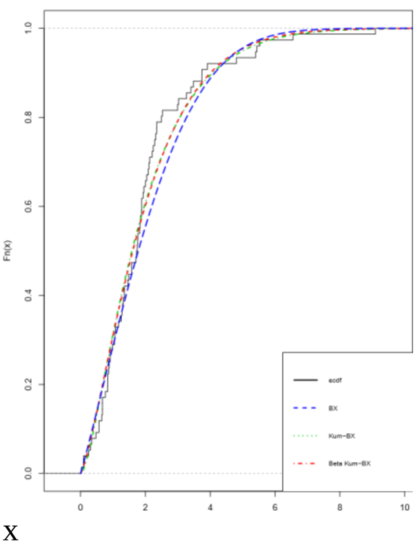


Figure 8: The empirical distribution for Life of Fatigue for Kevlar Right Skewed Dataset.

This proves that KBX distribution is a good example for modeling right skewed datasets. In this case, the likelihood ratio (LR) statistic for testing H_o versus H_1 is $\omega^* = 2[L(1,1,\vartheta,\tau) - L(\phi,\psi,\vartheta,\tau)]$, where the $L(\phi,\psi,\vartheta,\tau)$ is the log-likelihood statistic for the new KBX distribution and $L(1,1,\vartheta,\tau)$ is the log-likelihood statistic for the BX distribution respectively. The statistic ω is asymptotically ($n \to \infty$) distributed as X_k^2 where k is the number of parameters specified underH_o. The LR test rejects H_o if the $\omega > \eta_\rho$ where η_ρ denote the upper 100ρ% point of the X_k^2 distribution. Given that, $\omega^* = 2[(127)-(121)] = 12$, the LR of the null hypothesis tests of BX vs KBX H_o: all the model are not independently versus H_1 : all the model are independently distributed,. Based on the data $\omega = 12.0 > 5.991 = \chi_{2;0.05}$, we therefore reject the null hypothesis.

5.8. Strengths of 1.5 cm Glass Fibers (Left Skewed Dataset)

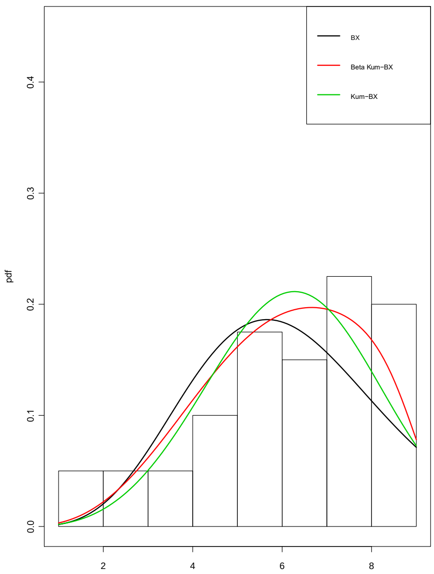
The second dataset consists of 63 observations of the strengths of 1.5 cm glass fibers, originally obtained by workers at the UK National Physical Laboratory. 280 On the other side they fail to include the units of measurements in the work". The data have also been analyzed by [22, 40].

The data set is given in Table (6), presents the MLE estimates for the parameters and the values of $-2LL(\theta)$,(K-SM), AIC, AICc and BIC statistics.

The KBX, BKBX and BX distributions are applied to fit the data set. The values in Table (6), indicates that the KBX is a strong competitor and also with the highest *p*-value based on the goodness-of-fit to BKBX and BX distributions used here for fitting the data.

The KBX provides the best fit because it has the smallest value of statistics for the strengths of 1.5 cm glass fibers data followed by the KBX distribution. 290 This data set has long tail to the left that is left-skewed or negatively-skewed. This example suggests that KBX fits very well the left-skewed data. Figure (9), provides the histogram while, Figure (10), provides the empirical CDF, showing KBX closer to it over BKBX and BX, this proves the flexibility of KBX is a

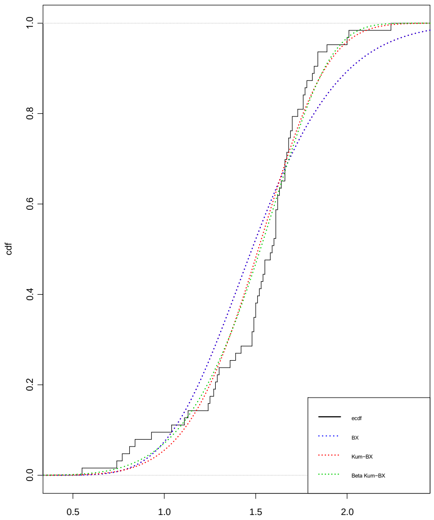
Fitted Densities



Strengh of 1.5 glass fiber

Figure 9: The histogram for the Strengths of 1.5 cm Glass Fibers Left Skewed.

Empirical Distribution



The strengths of 1.5 cm glass fibers dataset.

Figure 10: The empirical distribution for the Strengths of 1.5 cm Glass Fibers Left Skewed Dataset.

Table 6: The ML estimates, -2log-likelihood, AIC, AICc, BIC and (KSM) for the strengths of 1.5 cm glass fibers dataset

Model	MLE	$-2LL(\theta)$	AIC	AICc	BIC	K-SM	*p*-value
KBX	ϕ =1.611 ψ =0.564 ϑ= 4.748 τ =1.231	14.999	40.1	41.8	49.7	0.167	0.893
BKBX	v =0.342 κ = 4.217 ϕ =19.187 ψ= 6.675 ϑ =0.475 τ= 0.582	23.929	41.5	43	54.4	0.155	0.736
BX	ϑ=5.486 τ =0.987	32.905	51.9	52.1	56.1	0.215	0.623

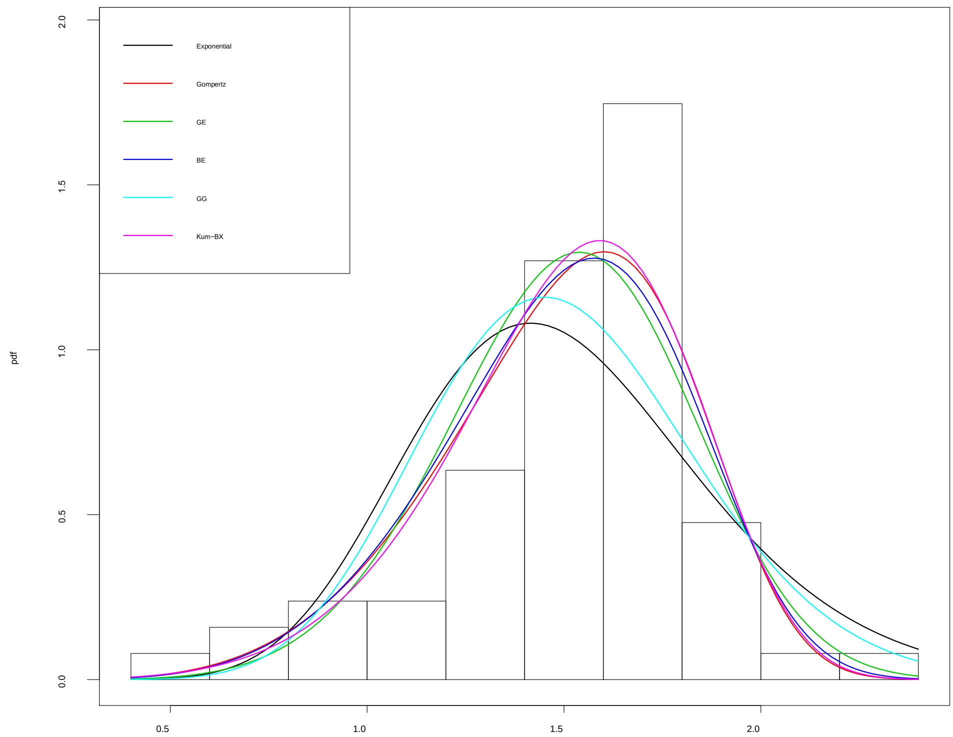
stronger with highest peak showing the histogram and empirical distribution
as it fits the strength of fiber glass left-skewed data perfectly better than the BKBX and the baseline BX distributions respectively.
Table (6), shows results for the fitted model for the strength data and the estimated results of criterion and the *p*-values as fitted based on the above dataset was left skewed. This proves that KBX distribution is a good example for modeling left skewed datasets. Given that, $\omega^* = 2[(32.905) - (14.999)] = 35.812$, the LR of the null hypothesis tests of BX vs KBX H_o: all the model are not independently versus H_1 : all the model are independently distributed,. The LR result indicates that the KBX is a very good for the strength of fiber glass dataset. Based on the data above $\omega = 35.812 > 5.991 = \chi_{2;0.05}$, we therefore
reject the null hypothesis. The LR result indicates that the KBX is a very good fit for the strength of fiber glass dataset.

5.9. **Nicotine Measurements (Approximate Symmetry Dataset)**
The following data is about 317 in number of the nicotine which was compiled in 2004. It was generated by the Federal Trade Commission which is an independent agency of the US government by [39]. Table (7), shows MLE estimate for the individual fitted model for the" Nicotine measurements" data and the estimated $-2LL(\theta)$, AIC, AICc, BIC and (K-SM) values with highest *p*-value, also the results obtained indicates clearly that KBX is flexible and vital to the existing models based on the nicotine measurement dataset.
Table 7: The ML estimates, -2log-likelihood, AIC, AICc, BIC and (KSM) for the Measurements on Nicotine dataset.

Model	MLE	$-2LL(\theta)$	AIC	AICc	BIC	K-SM	*p*-value
KBX	$\phi=0.216$ $\psi=0.247$ $\vartheta=0.035$	220.672	449.344	450.233	456.092	0.128	0.744
GG	$\psi=0.009$ $\gamma=0.0896$ $\tau=0.089$ $\varphi=0.026$	231.244	455.488	462.009	467.224	0.219	0.532
Gompertz	$\psi=0.109$ $\gamma=0.020$	235.331	474.662	474.917	478.486	0.401	0.213
BE	$\nu=0.524$ $\kappa=0.085$ $\psi=0.235$	238.110	482.239	482.702	480.976	0.572	0.302
GE	$\phi=0.902$ $\psi=0.0212$	242.375	484.271	483.026	481.595	0.663	0.021
Exponential	$\psi=0.022$	245.089	485.179	488.563	485.091	0.773	0.152

The Figure (11), for the plot of densities which compares the models with the histogram proves that KBX is closer to the histogram peak point than the existing among the generalized exponential, Beta exponential, Gompertz, gen

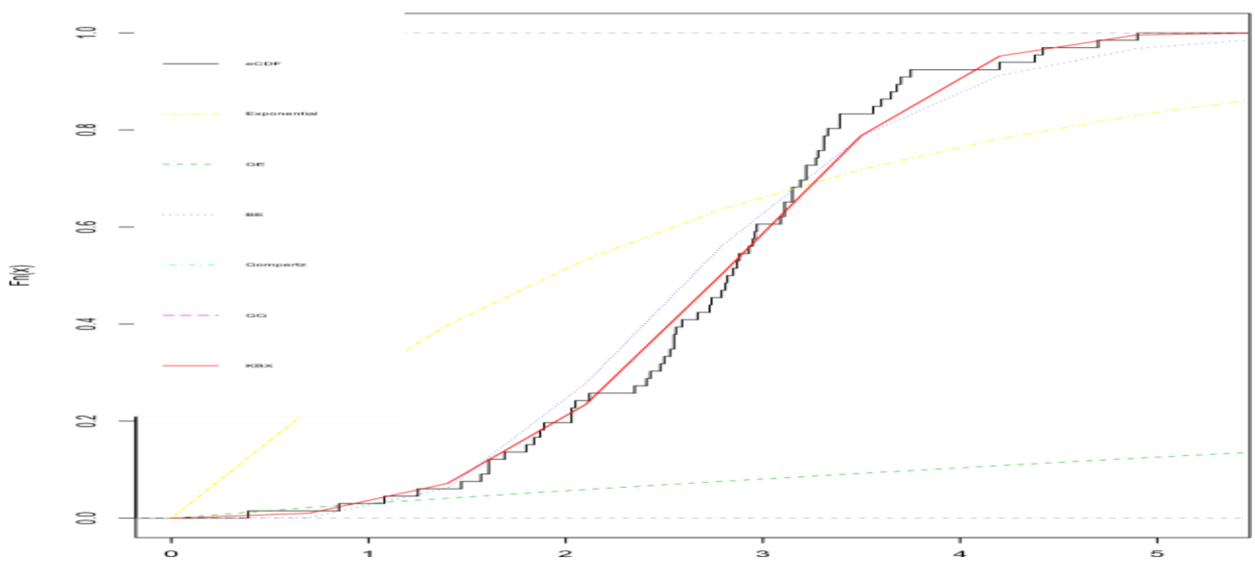
Fitted Densities



Nicotine Measurement

Figure 11: The histogram for the Plot of KBX with existing models for the Nicotine Measurements Approximate Symmetry Dataset.

Empirical Distribution



x

Figure 12: The empirical distribution for the plot of KBX with existing models for the Nicotine Measurements Approximate Symmetry Dataset.

eralized Gompertz, exponential distributions respectively. From all indication looking at the empirical CDF in Figure (12), for the comparison it clearly shows that based on this datasets KBX distribution is close to the empirical CDF. We therefore recommend that KBX is very good in modeling nicotine measurement data which is approximate symmetry (normally) datasets.
the likelihood ratio (LR) statistic for testing H_o versus H_1 is $\omega^* = 2[L(1,1,\vartheta,\tau)-L(\phi,\psi,\vartheta,\tau)]$, where the $L(\phi,\psi,\vartheta,\tau)$ is the log-likelihood statistic for the new
KBX distribution and $L(1,1,\vartheta,\tau)$ is the log-likelihood statistic for the all the non-nested models respectively.

Given that,

$$\text{KBX vs GG}, \omega^* = 2[(231.244) - (220.672)] = 21.1,$$
$$\text{KBX vs G}, \omega^* = 2[(235.331) - (220.672)] = 29.3,$$
$$\text{KBX vs BE}, \omega^* = 2[(238.110) - (220.672)] = 34.9,$$
$$\text{KBX vs GE}, \omega^* = 2[(242.375) - (220.672)] = 43.4,$$
$$\text{KBX vs E}, \omega^* = 2[(245.089) - (220.672)] = 48.8.$$

Where, $1 - \chi^2(\omega^*, \text{number of df of new model} - \text{nonnestedmodel})$

$=1 - \chi^2(\omega^*, n_1 - n_2)$, to obtain the *p*-value, using R software respectively. This proves that KBX distribution is a good example for modeling symmetry datasets. Also, the LR test with the hypothesis H_o: all the model are not independently versus H_1 : all the model are independently distributed. Based on the data, a comparison will be based on the new model KBX and the existing ones are as follows: $\omega* = \chi^2_{cal} > \chi^2_{tab} = \chi_{n1-n2;0.05}$, at 0.05% level of significance we therefore reject the null hypothesis and conclude that base on the result of comparison the KBX proves to be the best model.

6. Conclusion

We introduced a new model called Kumaraswamy Burr-Type X (Kum-BX) with four parameters $(\phi,\psi,\vartheta,\tau)$ that extends the Kumaraswamy-G family, and Burr-Type X distributions by method of beta Kumaraswamy-G family which was proposed by [4]. Kum-BX serve as an alternative to Kumaraswamy-Weibull model, which is very flexible distribution that has increasing, decreasing and bathtub shapes in the hazard function. We obtain the distributional properties like: P.D.F, C.D.F, Hazard function and their expansions. Also, the statistical properties like: Renyi entropy, quantile function, Browley skewness, Moors kurtosis, etc. were also obtained. A simulation study at different sample sizes with parameter values was done to validate and compare the mean errors where increasing the sample size decreases the error. The parameters were estimated by using M.L.E and Bayes methods applying real data set by using the goodness of fit is clarified. This new model allocates a better fitness, flexibility and vital to use in many areas when the data follows and suits a Kumaraswamy Burr Type X distribution than some its sub and existing models and it is very good model for right skewed data. We assume that the new model and its generalizations may draw a valid recommended attention to several areas such as; agriculture, engineering, medical, survival/reliability analysis and economics among others.

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