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Beta Kumaraswamy Burr Type X Distribution and Its Properties

Umar Yusuf Madaki[a,b,*], Mohd Rizam Abu Bakar[a,c], Laba Handique[d]

[a]Department of Mathematics, Faculty of Science, Universiti Putra Malaysia, MALAYSIA
[b]Department of Mathematics and Statistics, Faculty of Science, Yobe State University, NIGERIA
[c]Laboratory of Computational Statistics and Operations Research, Institute for Mathematical Research, Universiti Putra Malaysia, MALAYSIA
[d]Department of Statistics, Faculty of Science, Dibrugarh University, INDIA

*Corresponding author email: uymadaki84@gmail.com

Abstract We proposed a so-called Beta Kumaraswamy Burr Type X distribution which gives the extension of the Kumaraswamy-G class of family distribution. Some properties of this proposed model were provided, like the expansion of densities and quantile function. We considered the Bayes and maximum likelihood methods to estimate the parameters and also simulate the model parameters to validate the methods based on different set of true values. Some real data sets were employed to show the usefulness and flexibility of the model which serves as generalization to many sub-models in the fields of engineering, medical, survival and reliability analysis.

Keywords: Bayesian, Beta Kumaraswamy-G, Burr Type X, Kumaraswamy-G, quantile function, maximum likelihood estimation, inverse CDF method of simulation

1. Introduction

Here, we provide a brief explanation on the Beta distribution which has been receiving more attention in the area of statistical modeling over a century [1]. The beta distribution has PDF and CDF defined as:

$$f^{Beta}(t; \nu, \kappa) = \frac{1}{B(\nu, \kappa)} t^{(\nu-1)} (1-t)^{(\kappa-1)}, t \, \epsilon (0, \tag{1}$$

$$F^{Beta}(t; \nu, \kappa) = I_t(\nu, \kappa) \quad t \, \epsilon (0, 1). \tag{2}$$

where the shape parameters, ν and κ, are positive. Its functions are unimodal, uni-anti modal, bathtub or constant depending on the parameter values. Beta models are adaptable in-terms of versatility and this flexibility motivates its usage in many applications. On the other side, we have Kumaraswamy (Kum) distribution [2], which fails to accept the fact that Beta model does not fit hydrological data. The PDF and CDF of the Kum Model, Kum(ϕ, ψ), is given as:

$$f^{Kum}(t; \varphi, \psi) = \varphi, \psi t^{(\varphi-1)} (1-t^\varphi)^{(\psi-1)} \quad t \, \epsilon (0, 1) \tag{3}$$

$$F^{Kum}(t; \varphi, \psi) = 1 - (1-t^\varphi)^{(\psi)} \quad t \, \epsilon (0, 1). \tag{4}$$

1.1. Beta-Generated (Beta-G) Family of Distribution

Beta-G family was proposed by [3]and [31] is a very versatile and rich class of generalized distributions. It recieves a quite well response over the recent years. It has a PDF and CDF as follows:

$$f^{BG}(t; \nu, \kappa) = \frac{1}{B(\nu, \kappa)} f(t) F(t)^{(\nu-1)} S(t)^{(\kappa-1)}, \tag{5}$$

$$F^{BG}(t; \nu, \kappa) = \frac{1}{B(\nu, \kappa)} \int_0^{F(t)} W^{(\nu-1)} (1-W)^{(\kappa-1)} dW = I_{F(t)}(\nu, \kappa). \tag{6}$$

where $S(t)$ is the survival function of Beta-G distribution. Some of the well-known works related to Beta-G family of distributions received an impact for a given distribution with PDF and CDF of Beta-G proposed by [3], [4] and [5] family of distribution.

1.2. Kumaraswamy-G (Kum-G) Family of Distribution

Kumaraswamy-G family received a quite well attention in the recent years in the area of applied statistics proposed by [6], this family called Kum-G family motivate us in which we applied the method which proposed by [7] in developing our model having PDF and CDF define respectively as:

$$fKum-G(t;\phi,\psi) = \phi\psi g(t)G(t)(\phi-1)[1-G(t)\phi](\psi-1), \qquad (7)$$
$$F^{Kum-G}(t;\phi,\psi) = 1-[1-G(t)^\phi]^\psi, \qquad (8)$$

where, $t > 0, g(t) = \frac{d}{dt}G(t)$ and $\phi, \psi > 0$, denotes the shape parameters on the notion for other parameters in the baseline distribution. Kumaraswamy-G class by [6], received a quite well attention to researchers in the last few decades in the literature.

1.3. Beta Generated Kumaraswamy-G (Beta Kum-G) Family of Distribution

Recently [8]-[32], proposed this new family of distribution called Beta Generated Kumaraswamy-G (Beta Kum-G) distribution having $\theta = p,q$ ϕ,ψ,ϑ and τ where they adopt the method of Beta-Generator proposed by [3] and reviewed by [4]-[5]. This new family called beta Kum-G family motivate us in which we applied the method by [7], in developing our model. It has a PDF and CDF given as:

$$f^{BKum-G}(t;\nu,\kappa,\varphi,\psi) = \frac{1}{B(\nu,\kappa)}\varphi\psi g(t)G(t)^{(\varphi-1)},$$
$$\times [1-G(t)^\varphi]^{(\psi\kappa-1)}[1-\{1-G(t)^\varphi\}^\psi]^{(\nu-1)}, \qquad (9)$$
$$F^{BKum-G}(t;\nu,\kappa,\varphi,\psi) = I_{1-[1-G(t)^\varphi]^\psi}(\nu,\kappa). \qquad (10)$$

This new family can also be referred to as the new generalized Kumaraswamy-G (GKum-G) family of distribution.

1.4. Burr Type X (BX)

This model was proposed by [9] and revisited by [10]-[11], who contribute a lot to family of continuous distributions and played a vital role in medical, survival and reliability analysis. Our first motivation arises to the failure or hazard rate function having some important features including the Burr-Type X (BX) property which exhibits the fitting of engineering and medical data sets, more especially big data, as it was recently observed by [12], that BX model is very effective and versatile in modeling reliability strength and lifetime data sets. They also thought BX model can be exponentiated Rayleigh (ER) or generalized Rayleigh (GR), but they prefer calling it BX model due to its suitability to Burr family of distribution and properties. They observed that the GR or BX model with two-parameter also has a quite common properties with gamma, generalized exponential and Wei-bull distributions respectively. Although, the BX model density and cumulative distribution functions have a simple close form and it's also having a convenient and flexible feature in modeling censored (incomplete) data, unlike gamma, GE and Weibull distributions. The two-parameter BX has a monotonically increasing and decreasing hazard function features, which can be used for practical aspects in statistical distribution and modeling of applications. Recently, authors have been studying BX model due to its ability and flexibility in modeling reliability data sets such as, [13] whom proposed the two parameter Burr type X distribution which authors like; [14]and [15] made some extension. The Burr type X distribution can be used in modeling general lifetime data. Based on this model distribution in [10], whose PDF and CDF are given by:

$$fBX(t;\vartheta,\tau) = 2\vartheta\tau 2te^{-(\tau t)^2}\{1-e^{-(\tau t)^2}\}(\vartheta-1), \qquad (11)$$
$$FBX(t;\vartheta,\tau) = \{1-e^{-(\tau t)2}\}\vartheta. \qquad (12)$$

These two Figure (1) and Figure (2), were originally plotted by [11], showing the flexibility (left-skewed, right-skewed and symmetry), shapes (bathtub and increasing) respectively. The two parameters BX or GR model in equation (11) above, implies $BX(\vartheta,\tau)$, if $\vartheta = 1$, BX distribution reduces to a well-known one parameter Rayleigh distribution. Where ϑ and τ are the shape and scale parameters respectively. The two parameters BX or GR model in equation 11.
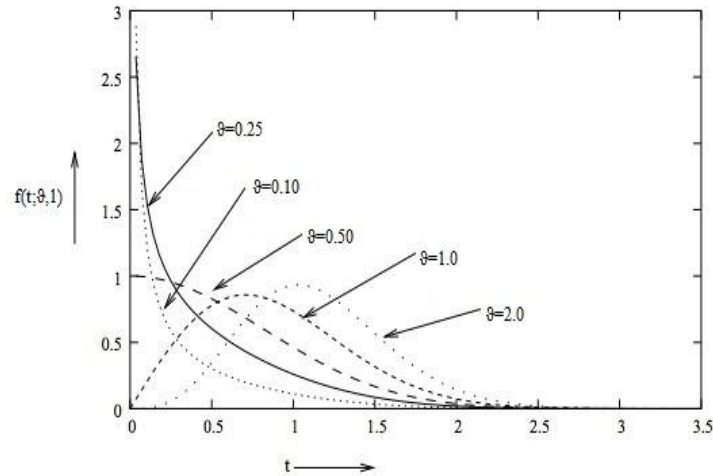
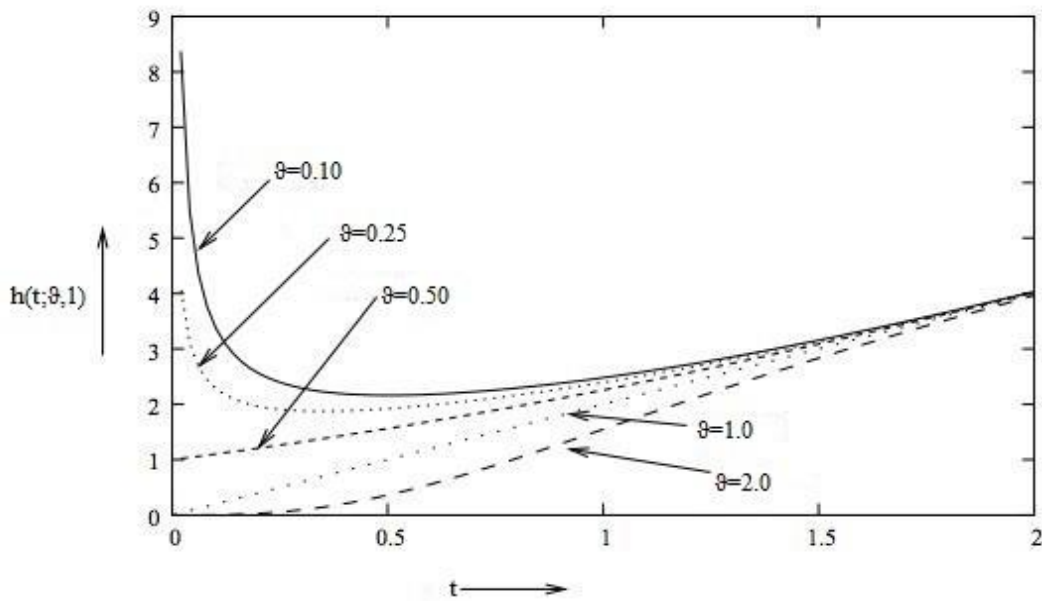**Figure 1:** Plot of the Burr Type X probability density function for the shape parameter ϑ.

Figure 2: Plot of the Burr Type X hazard function for the shape parameter ϑ

above, implies $BX(\vartheta, \tau)$, if $\vartheta = 1$, BX distribution reduces to a well-known one parameter Rayleigh distribution. If $\vartheta \neq 1$, this plays the role of τ, which is the scale parameter. On the other hand, if $\vartheta \leq \frac{1}{2}$, therefore the PDF of BX model will be decreasing function of the model while for $\vartheta > \frac{1}{2}$, it proves that it is a unimodal right skewed function. If it has a mode written as $\frac{t_o}{\tau}$, where t_o is called the non-linear systems of equation given as:

$$1 - 2t^2 - e^{-t^2}(1 - 2\vartheta t^2) = 0.$$

The above mode of BX model shows clearly that it is decreasing function of the scale parameter τ and also the increasing function of ϑ respectively. The PDF figures at different forms resemble that of Weibull and gamma functions. Moreover, the median of the BX model occurs when the given quantile at:

$$\left[-\frac{1}{\tau} \ln(1 - \frac{1}{2})^{\frac{1}{\vartheta}} \right]^{\frac{1}{2}}.$$

This also shows that the non-increasing form of τ, and the non-decreasing form of ϑ. The remaining part of the section were organized as follows. In section 2, we provide the distributional properties, the expansions for the PDF and CDF. In section 3, the expansion of densities, sub-models. In Section 4, we obtain 0 the moments, moment generating function and the order statistics, Re nyi entropy, the quantile function, Skewness and Kurtosis and maximum likelihood estimation. In section 5, application of real data and comparison of the new model with some sub-models was done and follow by the conclusion in section **??** respectively.

2. Distributional Properties

2.1. Beta Kumaraswamy Burr Type X (Beta Kum-BX) Distribution

In this section, we present our new model Beta Kum-BX with six parameters as the properties. Our second motivation, lies within the wide usage of Kum-G family define in equation (7) and the baseline distribution known as the BX define in equation (11) together with the property of beta-G family define in equation (5) and (6) above due the relative flexibility and capability in modeling agriculture, engineering and medical datasets in respect to the model suitability at different tractable or complex situations. However, we were motivated to introduced this new model called Beta Kumaraswamy Burr Type X (Beta KumBX) with six parameters $\theta = \nu, \kappa, \phi, \psi, \vartheta$ and τ by confounding equation (9) and (11) and also equation (10) and (12) by the methods of beta-G generator proposed by [8] and thereby Burr Type X as the baseline distribution which we obtain the probability density function PDF given as:

$$
\begin{aligned}
f^{BKBX}(t;\theta) =& \frac{2}{B(\nu,\kappa)}\varphi\psi\vartheta\tau^2 te^{-(\tau t)^2}\{1 - e^{-(\tau t)^2}\}^{(\vartheta\varphi-1)} \\
& \times [1 - \{1 - e^{-(\tau t)^2}\}^{\vartheta\varphi}]^{\psi\kappa-1} \\
& \times [1 - [1 - \{1 - e^{-(\tau t)^2}\}^{\vartheta\varphi}]^{\psi}]^{\nu-1}.
\end{aligned}
\tag{13}
$$

We now show that the $\int_0^t f(t)dt = 1$. Therefore from equation (13), above we simplify the model and proof the theorem.

Hence, let $M = [1 - \{1 - e^{-(\tau t)2}\}^{\vartheta\phi\psi}]$ and $d_M = 2\phi\psi\vartheta\tau 2te-(\tau t)^2\{1 - e-(\tau t)^2\}(\vartheta\phi-1)$

Therefore, equation (13), becomes more like a complete beta function given as;

$$
\int_0^t f(t)dt = \frac{1}{B(\nu,\kappa)}\int_0^t M^{(\kappa-1)}(1 - M)^{(\nu-1)}d_M,
$$

were, $\int_0^t M^{(\kappa-1)}(1 - M)^{(\nu-1)}d_M = B(\nu,\kappa)$

Hence, $\frac{1}{B(\nu,\kappa)} \times B(\nu,\kappa) = 1$.

The corresponding cumulative distribution function, survival function $S(t)$, hazard or failure rate function $h(t)$ and cumulative hazard function $H(t)$ respectively.

$$
F^{BKBX}(t;\theta) = I_{[1-(1-e^{-(\tau t)^2})^{\vartheta\varphi}]^{\psi}}(\nu,\kappa),
\tag{14}
$$

$$
S^{BKBX}(t;\theta) = 1 - F^{BKBX}(t),
\tag{15}
$$

$$
= 1 - I_{1-\{(1-e^{-(\tau t)^2})^{\vartheta\varphi}\}^{\psi}}(\nu,\kappa),
\tag{16}
$$

$$
\begin{aligned}
h^{BKBX}(t;\theta) =& \frac{2}{\{(1-e^{-(\tau t)^2})^{\vartheta\varphi}\}^{\psi}B(\nu,\kappa)}, \\
& \times \varphi\psi\vartheta\tau^2 te^{-(\tau t)^2}\{1 - e^{-(\tau t)^2}\}^{(\varphi-1)} \\
& \times [1 - \{1 - e^{-(\tau t)^2)^{\vartheta\varphi}}\}]^{\psi\kappa-1}, \\
& \times [1 - [1 - \{1 - e^{-(\tau t)^2}\}^{\vartheta\varphi}]^{\psi}]^{\kappa-1}
\end{aligned}
\tag{17}
$$

$$
H^{BKBX}(t;\theta) = -\ln\left[\{(1-e^{-(\tau t)^2})^{\vartheta\varphi}\}^{\psi}\right].
$$

Based on the relative usage of Kumaraswamy and Burr Type X baseline distributions we were motivated to propose this six parameters Beta Kum-BX due to the wideness of the BX with two parameters which generalizes (Rayleigh) and it provides a continuous function among the twelve Burr family of distribution which makes it suitable in modeling several complex situations. Also, the model transformation in equation (14) are not submissive meaning not mostly tractable just

like the Weibull distribution where the skewness and kurtosis at different shapes were flexible with monotonically non-decreasing and non-increasing hazard functions.

Our last motivation was noted from the above CDF of the new model equation (14), $FBetaKum-BX(t,\theta)$ m_1Y as $t \to 0$ also $1 - FBetaKum-BX(t,\theta) \sim m_2 \exp-(\tau t)^2$ as $t \to \infty$ where m_1 and m_2 denotes the model constraints. This new model BKBX encloses a lower tail performance for the exponentiated Rayleigh model and the upper tail acts like the Burr-Type X model. Kumaraswamy Burr-Type X which are the (Kumaraswamy and Burr-Type X) motivates us with its great varieties of fitting failure, engineering (reliability), medicine and agriculture etc. This six-parameter model encompasses many sub-models with great flexibility which in their bathtub shapes, monotonically non-increasing and decreasing failure function and also has some common properties with Kumaraswamy-Weibull by [18] which fit more perfectly well than the reduce modified Kumaraswamy Burr type X model relating to flexibility and suitability towards multivariate and censored data respectively.

Figure (3) and Figure (4), describes the shapes of the probability density function and cumulative distribution function functions for the given parameter values. These functions represent different kind of forms depending on choosing values of BKBX model parameters. We noticed that the additional shape parameters allow for high level of flexibility.

2.2. Shapes of the BKBX Distribution
On the other hand, Figure (5) and Figure (6) below, shows the failure rate or hazard function of BKBX model increasing and decreasing or bathtub shapes.
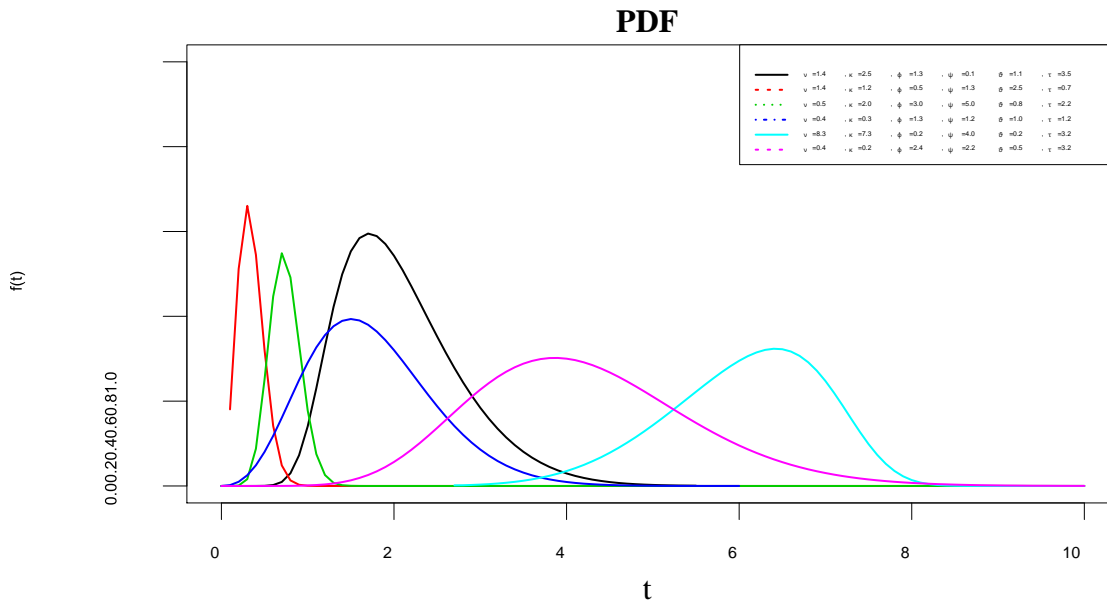
Figure 3: Plot of the BKBX probability density function (PDF) for different parameter values of $v, \kappa, \phi, \psi, \vartheta$ and τ.

This new model with six parameters is more flexible than Kumaraswamy Burr Type X and Burr Type X distribution with four and two parameters due to flexibility of Kumaraswamy distribution with two parameters, that leads more smooth and vital. This new model will be useful in modeling and analyzing real life, censored and uncensored data in medical, engineering, pure science and agricultural areas.
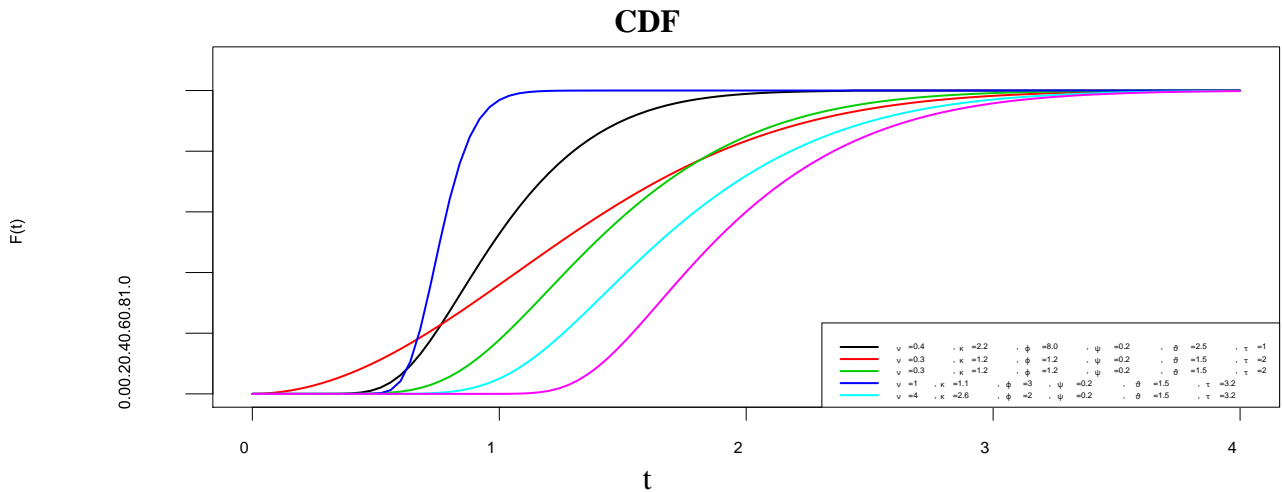
Figure 4: Plot of the BKBX cumulative distribution function (CDF) for different parameter values of $\nu, \kappa, \phi, \psi, \vartheta$ and τ.

3. Expansion of PDF and CDF

We provide expansion of the PDF of BKBX distribution in terms of infinite weighted sum. We obtain some structural properties based on the pdf expansions such as moment, moment generating function, mean division and others. The expansion is being done by using the equation of the PDF from equation (13). According to [6], if $|W| < 1$ and $\omega > 0$ is a real non-integer, we have the series representation:

$$(1 - W)^{\omega - 1} = \sum_{j=0}^{\infty} (-1)^j \binom{\omega - 1}{j} W^j \tag{18}$$

If ω is an integer number, the series representation for finding CDF expansion is:

$$(1 - W)^{\omega - 1} = \sum_{j=0}^{\omega - 1} (-1)^j \binom{\omega - 1}{j} W^j$$

By using the expansion of equation (18) and $\omega > 0$ real non-integer, By using the binomial expansion in equation (13), above which is the density function of BKBX distribution with six parameters, $\theta = (\nu, \kappa, \phi, \psi, \vartheta, \tau)$ and $\theta_2 =$

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Hazard

(a) Hazard Function 1.

Hazard

(t)

(b) Hazard Function 2.

(t)

(a) Hazard Function 3.

(t)
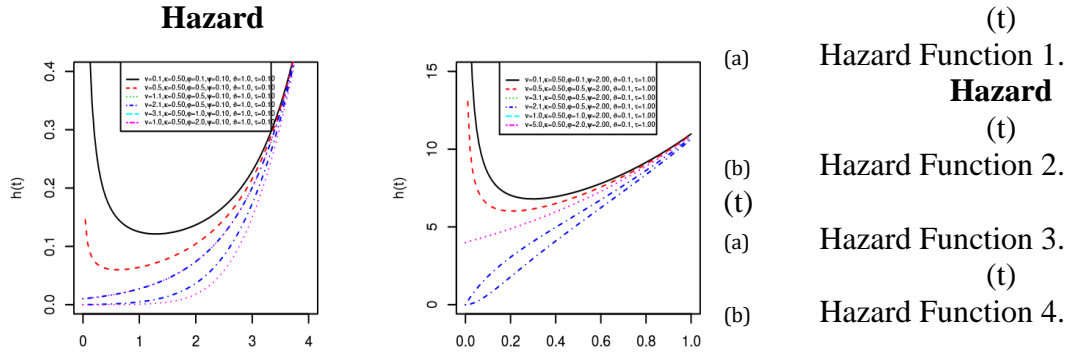
(b) Hazard Function 4.

Figure 5: Plot of the BKBX Hazard function for different parameter values of $v, \kappa, \phi, \psi, \vartheta$ and τ.
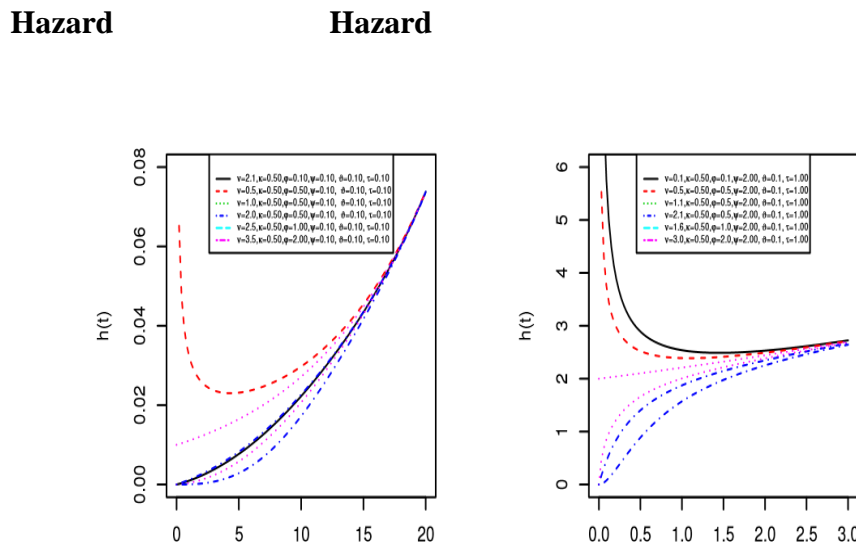
Hazard   **Hazard**

Figure 6: Plot of the BKBX Hazard function for different parameter values of $v, \kappa, \phi, \psi, \vartheta$ and τ.

$(\phi, \psi, \vartheta, \tau)$. We denote the following:
$M_i = 1 - e^{-(\tau t)^2}$,
$R_i = [1 - (M)^{\vartheta\phi}]$.

$$f^{BKBX}(t;\theta) = \frac{2}{B(\nu,\kappa)}\varphi\,\psi\,\vartheta\,\tau^2 t e^{-(\tau t)^2}[M_i]^{(\vartheta\,\varphi-1)}[R_i]^{\psi\kappa-1},$$

$$\times \sum_{j=0}^{\nu-1}(-1)^j\binom{\nu-1}{j}[1-(1-e^{-(\tau t)^2})^\varphi]^{\psi j},$$

$$=\sum_{j=0}^{\nu-1}(-1)^j\binom{\nu-1}{j}\acute{\psi}_j f^{KBX}(t;\theta_2(j+n)), \qquad (19)$$

$$\acute{\psi}_j = \frac{(-1)^{j+1}}{B(\nu,\kappa)(j+n)}\binom{\nu-1}{j}.$$

Were,
Adjusting further we obtained from Equation (19).

$$\begin{aligned}(20) \quad &= \sum_{j=0}^{\nu-1} \acute{\psi}_j \frac{d}{dt} F^{KBX}(t; \theta_2(j+n))] \\ &= \sum_{j=0}^{\nu-1} \acute{\psi}_j \frac{d}{dt} F^{KBX}(t; \theta_2)]^{(j+n)},\end{aligned}$$

$\nu-1$

=fKBX(t;θ2)Xψ´jFKBX(t;θ2)](j+n−1).　(21)
j=0

Where, $\psi'_j = \psi_j(j + n)$ Alternatively, we can expand the PDF also used by [6]:

$\nu-1$

BKBX　　　　　KBX　　　　　　　X ´hFKBX(t;θ2)i(j+n−1),　　　　　　(22)
f　　　　(t;θ) =f　　　(t;θ2)　　ψj
j=0 ν−1

$$\begin{aligned}&= f^{KBX}(t; \theta_2) \sum_{j=0}^{\nu-1} \acute{\psi}_j \sum_{l=0}^{j+n-1} \binom{j+n-1}{l} (-1)^l \left[F^{KBX}(t; \theta_2) \right]^l\end{aligned}$$

j=0 ν−1
KBX
=f　　　(t;θ2)

$$= f^{KBX}(t; \theta_2) \sum_{j=0}^{\nu-1} \eta_l \left[F^{KBX}(t; \theta_2) \right]^l,$$

=fKBX(t;θ2)Xψ´jh1 −
FKBX(t;θ2)i(j+n−1),

X{ψ´j(j + n)}hFKBX(t;θ2)i(j+n−1),

(23)

$$l = \acute{\psi}_j \sum_{l}^{j+n-1} .(-1)^l \binom{j+n-1}{l}$$

Where, η.
=0

Also we can expand the CDF with the following result using an incomplete beta function also provided by [5].

$$I_y(\varphi, \psi, \vartheta, \tau) = \frac{2y^{\vartheta\varphi}}{B(\nu, \kappa)} \sum_{\kappa=0}^{\infty} \binom{\psi-1}{\kappa} \frac{(-1)^\kappa y^\kappa}{\tau(\varphi+\kappa)}. \qquad (24)$$

Also, on the other hand, we have the CDF of BKBX as:
FBKBX(t;θ) = 1 − [1 − (1 − e−(τt)2)ϑϕ]ψ](v,κ).
Using Equation (24), above we obtained:

$$\begin{aligned}&= \frac{2}{B(\nu, \kappa)} \left[1 - [1 - (1 - e^{-(\tau t)^2})^{\vartheta \varphi}]^\psi \right]^\nu \sum_{i=0}^{\infty} \binom{\kappa-1}{i} \frac{(-1)^i}{\nu+i} \left[1 - [1 - (1 - e^{-(\tau t)^2})^{\vartheta \varphi}]^\psi \right]^i \\ &= \sum_{i=0}^{\infty} \frac{(-2)^i}{B(\nu, \kappa)(\nu+i)} \binom{\kappa-1}{i} \left[F^{KBX}(t; \theta_2)^{\nu+i} \right], \\ &= \sum_{i=0}^{\infty} \frac{(-2)^i}{B(\nu, \kappa)(\nu+i)} \binom{\kappa-1}{i} \sum_{j=0}^{\infty} \binom{\nu+i}{j} (-1)^j F^{KBX}(t; \theta_2)]^j,\end{aligned}$$

,

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$$= \sum_{i,j=0}^{\infty} \frac{(-2)^{i+j}}{B(\nu,\kappa)(\nu+i)} \binom{\kappa-1}{i} \binom{\nu+i}{j} \sum_{k=0}^{j} \binom{j}{k} (-1)^k \left[F^{KBX}(t;\theta_2) \right]^k \qquad , \tag{25}$$

By exchanging the indices j and k in the symbol, we get:

$$= \sum_{i,j=0}^{\infty} \sum_{k=0}^{j} \frac{(-2)^{i+j+k}}{B(\nu,\kappa)(\nu+i)} \binom{\kappa-1}{i} \binom{\nu+i}{j} \binom{j}{k} \left[F^{KBX}(t;\theta_2) \right]^k. \tag{26}$$

$$, \quad = \sum_{i,k=0}^{\infty} \sum_{j=k}^{\infty} \frac{(-2)^{i+j+k}}{B(\nu,\kappa)(\nu+i)} \binom{\kappa-1}{i} \binom{\nu+i}{j} \binom{j}{k} \left[F^{KBX}(t;\theta_2) \right]^k$$

Where, μ

$$= \sum_{k=0}^{\infty} \mu_k \left[F^{KBX}(t;\theta_2) \right]^k,$$

$$k = \sum_{i=0}^{\infty} \sum_{j=k}^{\infty} \frac{(-2)^{i+j+k}}{B(\nu,\kappa)(\nu+i)} \binom{\kappa-1}{i} \binom{\nu+i}{j} \binom{j}{k}.$$

On the other hand, the expansion of the CDF of BKBX can be derived similarly as provided by [33],

$$= \left[1 - \left(1 - (1-e^{-(\tau t)^2})^{\vartheta} \varphi]^{\psi} \right)(\nu,\kappa) \right] = \left[F^{BKBX}(t;\theta) \right]$$

$$= \sum_{m=\nu}^{\nu+\kappa-1} \binom{\nu+\kappa-1}{m} \left[F^{KBX}(t;\theta_2) \right]^m \left[F^{KBX}(t;\theta_2) \right]^{(\nu+\kappa-1-m)},$$

$$= \sum_{m=\nu}^{\nu+\kappa-1} \binom{\nu+\kappa-1}{m} \left[F^{KBX}(t;\theta_2) \right]^m$$

$$\times \sum_{n=0}^{\nu+\kappa-1-m} \binom{\nu+\kappa-1-m}{n} \left[F^{KBX}(t;\theta_2) \right]^n.$$

$$= \sum_{m=\nu}^{\nu+\kappa-1} \sum_{n=0}^{\nu+\kappa-1-m} \binom{\nu+\kappa-1}{m} \binom{\nu+\kappa-1-m}{n} (-1)^n \left[F^{KBX}(t;\theta_2) \right]^{(m+n)}$$

$$= \sum_{m=\nu}^{\nu+\kappa-1} \sum_{n=0}^{\nu+\kappa-1-m} \eta_l \left[F^{KBX}(t;\theta_2) \right]^{m+n}. \tag{27}$$

Where, η
$$l = \binom{\nu+\kappa-1}{m} \binom{\nu+\kappa-1-m}{n} (-1)^n.$$

3.1. Some Special Sub-models

The BKBX with $\theta = (\nu, \kappa, \phi, \psi, \vartheta, \tau)$ has several special cases or sub models but among are few stated below:

1. When $\phi = \psi = 1$, the BKBX in Equation (13) above reduces to Beta Burr Type X with four parameters by [14].

Show that equation (13) reduces to Beta Burr Type X with four parameters

Let $\phi = \psi = 1$ in equation (13), given as;

$$f^{BKBX}(t;\theta) = \frac{2}{B(\nu,\kappa)} (1)(1) \vartheta \, \tau^2 t e^{-(\tau t)^2} \{ 1 - e^{-(\tau t)^2} \}^{(\vartheta(1)-1)}$$

$$\times [1 - \{1 - e^{-(\tau t)^2}\}^{\vartheta(1)}]^{(1)\kappa-1}$$

$$\times [1 - [1 - \{1 - e^{-(\tau t)^2}\}^{\vartheta(1)}]^{(1)}]^{\nu-1}.$$

We can as well reduce to BBX model as follows:

$$f^{BBX}(t; \nu, \kappa, \vartheta, \tau) = \frac{2}{B(\nu, \kappa)} \vartheta \tau^2 t e^{-(\tau t)^2} \{1 - e^{-(\tau t)^2}\}^{(\vartheta - 1)}$$
$$\times [1 - \{1 - e^{-(\tau t)^2}\}^\vartheta]^{\kappa - 1}$$
$$\times [1 - (e^{-(\tau t)^2})^\vartheta]^{\nu - 1}.$$

Hence,

$$f^{BBX}(t; \nu, \kappa, \vartheta, \tau) = \frac{2\vartheta \tau^2 t e^{-(\tau t)^2}}{B(\nu, \kappa)} [1 - (e^{-(\tau t)^2})]^{(\vartheta \nu - 1)}$$
$$\times [1 - \{1 - e^{-(\tau t)^2}\}^\vartheta]^{\kappa - 1}.$$

2.   When $v = \kappa = 1$, the BKBX in Equation (13) above reduces to Kumaraswamy Burr Type X (KBX) with four parameters from the family of Kumaraswamy-G proposed by [6] also a special case to KumaraswamyWeibull distribution proposed by [14] respectively.
Show that equation (13) reduces to Kumaraswamy Burr Type X with four parameters Let $v = \kappa = 1$ in equation (13), given as;

$$f^{BKBX}(t; \theta) = \frac{2}{B(1, 1)} \varphi \psi \vartheta \tau^2 t e^{-(\tau t)^2} \{1 - e^{-(\tau t)^2}\}^{(\vartheta \varphi - 1)}$$
$$\times [1 - \{1 - e^{-(\tau t)^2}\}^{\vartheta \varphi}]^{(\psi(1) - 1)}$$
$$\times [1 - [1 - \{1 - e^{-(\tau t)^2}\}^{\vartheta \varphi}]^\psi]^{1 - 1}.$$

Hence, by law of indices let $A = [1 - [1 - \{1 - e^{-(\tau t)2}\}^{\vartheta \phi}]^\psi]$ and $A^0 = 1$.

$$= \frac{2}{B(1, 1)} \varphi \psi \vartheta \tau^2 t e^{-(\tau t)^2} \{1 - e^{-(\tau t)^2}\}^{(\vartheta \varphi - 1)}$$
$$\times [1 - \{1 - e^{-(\tau t)^2}\}^{\vartheta \varphi}]^{(\psi - 1)}$$
$$\times [1 - [1 - \{1 - e^{-(\tau t)^2}\}^{\vartheta \varphi}]^\psi]^0.$$

We can as well reduces to KBX model as follows:
$fKBX(t; \phi, \psi, \vartheta, \tau) = 2\phi\psi \vartheta\tau2te-(\tau t)^2\{1 - e-(\tau t)^2\}(\vartheta\phi-1)$
$\times \{1 - (1 - e-(\tau t)^2)(\vartheta\phi)\}(\psi-1).$

3. When $\phi = \psi = 1, v = \kappa = 1$, the BKBX in Equation (13) above reduces to Burr Type X distribution with two parameters [12]-[13].
Show that equation (13) reduces to Burr Type X with two parameters Let $v = \kappa = \phi = \psi = 1$ in Equation (13), given as;

$$f^{BKBX}(t; \theta) = \frac{2}{B(1, 1)} \varphi \psi \vartheta \tau^2 t e^{-(\tau t)^2} \{1 - e^{-(\tau t)^2}\}^{(\vartheta \varphi - 1)}$$
$$\times [1 - \{1 - e^{-(\tau t)^2}\}^{\vartheta \varphi}]^{(\psi(1) - 1)}$$
$$\times [1 - [1 - \{1 - e^{-(\tau t)^2}\}^{\vartheta \varphi}]^\psi]^{1 - 1}.$$

Hence, by law of indices let $A = 1 - [1 - \{1 - e^{-(\tau t)2}\}^{\vartheta(1)}]$ and $A^0 = 1$, likewise, $B = [1 - \{1 - e^{-(\tau t)2}\}^{\vartheta(1)}]$ and $B^0 = 1$

$$= \frac{2}{B(1,1)} (1)(1) \, \vartheta \, \tau^2 t e^{-(\tau t)^2} \{1 - e^{-(\tau t)^2}\}^{(\vartheta(1)-1)}$$

$$\times [1 - \{1 - e^{-(\tau t)^2}\}^{\vartheta(1)}]^{(1-1)}$$

$$\times [1 - [1 - \{1 - e^{-(\tau t)^2}\}^{\vartheta(1)}]^{(1)}]^0.$$

We can as well reduces to BX model as follows: $fBX(t;\vartheta, \tau) = 2\vartheta\tau 2te^{-(\tau t)^2}\{1 - e^{-(\tau t)^2}\}(\vartheta-1)$.

4. When $v = \kappa = \phi = \vartheta = \tau = 1$, the BKBX in Equation (13) above reduces to Rayleigh distribution with one scale parameter by [19].

Show that equation (13) reduces to Rayleigh with with one scale parameter. Let $v = \kappa = \phi = \vartheta = \tau = 1$ in equation (13), given **as;**

$$= \frac{2}{B(1, 1)} (1)(1)(1)\tau^2 t e^{-(\tau t)^2} \{1 - e^{-(\tau t)^2}\}^{((1)(1)-1)}$$

$$\times [1 - \{1 - e^{-(\tau t)^2}\}^{(1)(1)}]^{(1)(1)-1}$$

$$\times [1 - [1 - \{1 - e^{-(\tau t)^2}\}^{(1)(1)}]^{(1)}]^{(1)-1}.$$

We can as well reduces to R model from BX model substituting $\vartheta = 1$, by the law of indices let $C = \{1 - e^{-(\tau t)2}\}$ and $C^0 = 1$,:
$= 2(1)\tau 2te - (\tau t)^2\{1 - e - (\tau t)^2\}(0)$.

Hence, $fR(t;\tau) = 2\tau 2te - (\tau t)2$.

Figure (7) below shows the flow chart of all the sub-models of BKBX distribution at each level, where it provides how the six parameters reduces to its last submodel with one scale parameter Rayleigh (R) by [19] and one shape parameter Burr Type X (BX1) [9] respectively.
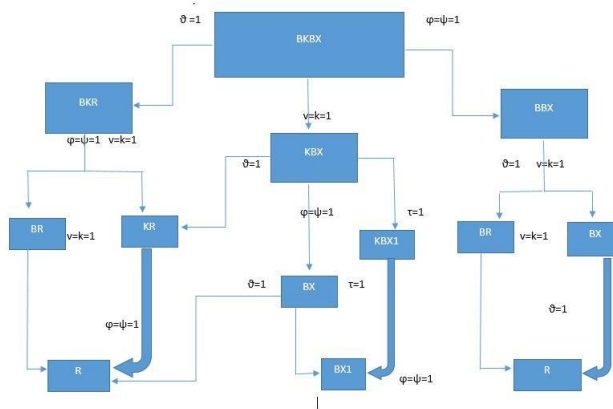

Figure 7: Sub models of BKBX Distribution

4. Probability Weighted r^{th} Moments

Probability weighted r^{th} moments (PWMs), It was initially introduced by [20], defined to be the expectation of some functions of a random variable x and y defined. The $(m,n,t))^{th}$ PWM of T is defined by:

$$\Gamma\varphi = \int_{-\infty}^{\infty} t^m F(t)^n [1 - F(t)]^r f(t) dt. \tag{28}$$

From the above equation (23) and equation (27), the sth moment of T can be written either as,

$$E(T^s) = \int_{-\infty}^{\infty} t^s f^{KBX}(t; \theta_2) dt,$$

$$= \sum_{j=0}^{\nu-1} \psi_j \int_{-\infty}^{\infty} t^s [F^{KBX}(t; \theta_2)]^{(j+\kappa-1)} f^{KBX}(t; \theta_2) dt,$$

$\nu-1$

$E(T^{s)} = X \psi_j \Gamma_{s,}0,j+\kappa-1,$

$j=0$ or,

$$= \sum^{j+\kappa-1} \eta_l \int_{-\infty}^{\infty} t^s [F^{KBX}(t; \theta_2)]^l f^{KBX}(t; \theta_2) dt,_{j=0}$$

$j+\kappa-1$

$E(Ts) = X \, \eta l \Gamma s,l,0.$

$j=0$

Where, $\Gamma_{s,l,0} = \int_{-\infty}^{\infty} t^s [F^{KBX}(t; \theta_2)]^l f^{KBX}(t; \theta_2) dt.$

$$\eta_l = \binom{\nu + \kappa - 1}{m} \binom{\nu + \kappa - 1 - m}{n} (-1)^n \qquad (29)$$

is the PWM of KBX $\theta_2 = (\phi, \psi, \vartheta, \tau)$ distribution. Therefore the moments of the BKBX $\theta = (\nu, \kappa, \phi, \psi, \vartheta, \tau))$ can be expressed in terms of the PWMs of KBX model. The PWM method can be generally be used for estimating parameters quantiles of generalized distributions.These moments have low variance and no severe biases, also they compare favorably with estimators obtained by maximum likelihood estimator. Proceeding as the above s^{th} moment of the rth order statistic $T_{r:q}$ in the random sample of size "n" [15]. From the order statistics in equation(31), we have,

$k+l$

$E(Trs:\kappa) = X \, \acute{\xi} y \, \Gamma s,0,z.$

$z=0$

Where, $\acute{\xi}_y = \sum_{j=0}^{k+\kappa-1} \sum_{k=0}^{\infty} \frac{(-1)^{y+1} \gamma_{j,k}}{y+1} \binom{k+j}{y} \underline{\qquad}.$

$\xi_y = \zeta_y(y+1).$

Where, $\acute{\psi}_j = \frac{(-1)^{j+1}}{B(\nu, \kappa)(j+n)} \binom{\nu-1}{j} \underline{\qquad}.$

Where, $\eta \, l = \binom{\nu + \kappa - 1}{m} \binom{\nu + \kappa - 1 - m}{n} (-1)^n.$

These follows an incomplete gamma function provided by [14].

$$\Gamma\varphi = \int_0^{\infty} e^{-y} y^{\varphi-1} dy$$

$$E(X^r) = \frac{\varphi\psi\vartheta\Gamma(r/2+1)}{B(\nu, \kappa)\tau^r} C_1 \frac{1}{(k+1)^{\frac{r}{2}+1}}$$

i $\qquad (30)$

Where, $C_1 = \sum_{i=0}^{\nu-1} \sum_{j=0}^{m(+n)-j} \sum_{k=0}^{\vartheta\varphi(j+1)} (-1)^{i+j+k} \binom{\nu-1}{i} \binom{m(i+n)-j}{j}.$

4.1. Moment Generating Function (MGF)

The MGF of BKBX distribution can be obtained and expressed in form of exponential Kum-G family of distribution from the results we obtained in the moments above using Equation 20 above,

$$M_T(s) = E[e^s t] = \int_{-\infty}^{\infty} e^s t f(t;\theta) dt,$$

$$= \sum_{j=0}^{\nu-1} \acute{\psi}_j \int_{-\infty}^{\infty} e^s t \frac{d}{dt} [F^{KBX}(t;\theta_2)]^{(j+n)} dt,$$

$$= \sum_{j=0}^{\nu-1} \acute{\psi}_j M_X(s),$$

Where, $M^X(s). = \int_{-\infty}^{\infty} e^s t \frac{d}{dt} [F^{KBX}(t;\theta_2)]^{(j+n)} dt.$

Where $M_X(s)$ is the MGF of a BKBX distribution.

4.2. Order Statistics

These two notable statisticians [10]and [11], proposed the procedure of deriving the order statistics of Burr Type X (BX) distribution. We adopted this method by assuming a random sample from a population $T_1, T_2,...,T_n$, from BKBX model with six parameters $\theta = (v, \kappa, \phi, \psi, \vartheta, \tau)$ and KBX model parameters, $\theta_2 = (\phi, \psi, \vartheta, \tau)$. We denote $T_{r:n}$ as the rth order statistics, given the PDF of $T_{r:n}$ can be express as:

$$f_{r:n}(t;\theta) = \frac{n!}{(r-1)!(n-r)!} f^{BKBX}(t;\theta) F^{BKBX}(t;\theta)^{(r-1)} \left[1 - F^{BKBX}(t;\theta)\right]^{\kappa-r},$$

Therefore, equation (30) reduces to this form by law of indices,

$$= \frac{n!}{(r-1)!(n-r)!} \sum_{k=0}^{\kappa-r} (-1)^j \binom{\kappa-r}{k} f^{BKBX}(t;\theta),$$

$$\begin{array}{cc} j+\kappa-1 & \infty \\ & \times X\, \eta_l [FBKBX(t;\theta)]l\ X\mu_k{}^h (FBKBX(t;\theta)) k^{ij+r-1}. \\ l{=}0 & k{=}0 \end{array}$$

By adopting the binomial expansion by [16], the PDF and CDF of the BKBX distribution. We obtained the PDF of the rth order statistics of is as follows:

$$f_{r:n}(t;\theta) = \frac{n!}{(r-1)!(n-r)!} \sum_{k=0}^{\kappa-r} (-1)^k \binom{\kappa-r}{k} \left(f^{KBX}(t;\theta_2) \sum_{j=0}^{k+\kappa-1} \eta_l \left[F^{KBX}(t;\theta_2)\right]^j \right)$$

$$\times \left(\sum_{k=0}^{\infty} d_{j+r-1} \left[F^{KBX}(t;\theta_2)\right]^k \right],$$

$$= \frac{n!}{(r-1)!(n-r)!} \sum_{k=0}^{\kappa-r} (-1)^k \binom{\kappa-r}{k} \left(f^{KBX}(t;\theta_2) \right.$$

$$\times \sum_{j=0}^{k+\kappa-1} \sum_{k=0}^{\infty} \eta_l d_{j+r-1} \left[F^{KBX}(t;\theta_2)\right]^{k+j} \Bigg),$$

$$f_{r:n}(t;\theta) = \sum_{y=0}^{k+j} \acute{\xi}_y f^{KBX}(t;\theta_2(y+1)),$$

$$\acute{\xi}_y = \sum_{j=0}^{k+\kappa-1} \sum_{k=0}^{\infty} \frac{(-1)^{y+1}\gamma_{j,k}}{y+1}\binom{k+j}{y},$$

$$\xi_y = \acute{\xi}_y(y+1),$$

$$\gamma_{j,k} = \frac{n!}{(r-1)!(n-r)!}\sum_{k=0}^{\kappa-r}(-1)^k\binom{\kappa-r}{k}\eta_l d_{j+r-1,k}$$

$$= f^{KBX}(t;\theta_2)\sum_{y=0}^{k+j}\frac{\xi_y}{y+1}\frac{d}{dt}\left[S^{KBX}(t;\theta_2)\right]^{y+1},$$

Hence, we obtained the rth order statistics of the BKBX distribution as:

(31)

.

4.3. Re´nyi Entropy

An entropy for a given random variable is called the measure of uncertainty used in many areas of research like: applied statistics and reliability in engineering. In the recent years, [33], modified and derived the general formula which was originally proposed by [34], over 5 decades ago. Moreover, [24], provides the expression of this entropies for models with more than one variates. It is given by definition:

$$I_R(\epsilon) = \left(1-\epsilon\right)^{-1}\ln\left(\int_{-\infty}^{\infty} f(t)^\epsilon dt\right). \tag{32}$$

4.4. Re´nyi Entropy

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Entropies are measures which quantified the randomness or diversity of a random variable X . They express the expected information uncertainty or content of a PDF. Entropy is a measure that provide important tools to analyze evolutionary processes over time (technical change) and to indicate the variety in the distributions at a particular moment of time. The large value of the R´enyi entropy indicates that there is greater uncertainty in the data. Where $\epsilon > 0$ and $\epsilon \neq 1$. Using binomial expansion also provided by [16], we can express it by denoting:

$$Mi = 1 - e-(\tau ti)2, \; Ri = [1 - (Mi)\vartheta\phi].$$

The R´enyi entropy for BKBX distribution can be found from equation (13), above as follows:

$$
\begin{aligned}
\int_0^\infty f(t)^\epsilon dt =& \left(\frac{2\varphi\,\psi\,\vartheta\,\tau^2}{B(\nu,\kappa)}\right)^\epsilon \int_0^\infty \left[t_i e^{-(\tau t_i)^2} M_i^{(\vartheta\varphi-1)} [1-R_i]^{\psi\kappa-1}\right]^\epsilon \\
& \times \left[1-R_i^\psi\right]^{\epsilon(\nu-1)} dt. \\
=& \left(\frac{2\varphi\,\psi\,\vartheta\,\tau^2}{B(\nu,\kappa)}\right)^\epsilon \sum_{i=0}^{(\nu-1)\epsilon} \binom{\epsilon(\nu-1)}{i}(-1)^i \int_0^\infty t_i^\epsilon e^{-\epsilon(\tau t_i)^2} \left[M_i^{(\vartheta\varphi-1)}\right]^\epsilon \\
& \times \left[1-R_i\right]^{\epsilon(\psi\kappa-1)} \left[1-R_i\right]^\psi dt. \\
=& \left(\frac{2\varphi\,\psi\,\vartheta\,\tau^2}{B(\nu,\kappa)}\right)^\epsilon \sum_{i=0}^{(\nu-1)\epsilon} \binom{\epsilon(\nu-1)}{i}(-1)^i \int_0^\infty t_i^\epsilon e^{-\epsilon(\tau t_i)^2} \left[M_i^{(\vartheta\varphi-1)}\right]^\epsilon \\
& \times \left[1-R_i\right]^{\epsilon(\psi\kappa-1)+\psi_i} dt. \\
=& \left(\frac{2\varphi\,\psi\,\vartheta\,\tau^2}{B(\nu,\kappa)}\right)^\epsilon \sum_{i,j=0}^{(\nu-1)\epsilon} \binom{\epsilon(\nu-1)}{j}\binom{\epsilon(\psi\kappa-1)+\psi_i}{i}(-1)^{i+j} \\
& \times \int_0^\infty t_i^\epsilon e^{-\epsilon(\tau t_i)^2} \left[M_i^{(\vartheta\varphi-1)}\right]^\epsilon dt.
\end{aligned}
\tag{34}
$$

Therefore, we obtained the R´enyi entropy by substituting equation (34), in equation (33) as follows:

$$
\begin{aligned}
I_R(\epsilon) =& \left(1-\epsilon\right)^{-1} \ln\left\{\left(\frac{2\varphi\,\psi\,\vartheta\,\tau^2}{B(\nu,\kappa)}\right)^\epsilon\right. \\
& \times \sum_{i,j=0}^{(\nu-1)\epsilon} \binom{\epsilon(\nu-1)}{j}\binom{\epsilon(\psi\kappa-1)+\psi_i}{i}(-1)^{i+j} \\
& \left. \times \int_0^\infty t_i^\epsilon e^{-\epsilon(\tau t_i)^2} \left[M_i^{(\vartheta\varphi-1)}\right]^\epsilon dt\right\}.
\end{aligned}
\tag{35}
$$

Thus, the R´enyi entropy is,

$$
\begin{aligned}
I_R(\epsilon) =& \left(1-\epsilon\right)^{-1} \ln\left(\frac{2\varphi\,\psi\,\vartheta}{B(\nu,\kappa)}\right)^\epsilon + \ln(2\tau) \\
& + \frac{1}{1-\epsilon} \ln\left\{\sum_{i,j=0}^{(\nu-1)\epsilon} \binom{\epsilon(\nu-1)}{j}\right. \\
& \times \binom{\epsilon(\psi\kappa-1)+\psi_i}{i}(-1)^{i+j} \\
& \left. \times \int_0^\infty t_i^\epsilon e^{-\epsilon(\tau t_i)^2} \left[M_i^{(\vartheta\varphi-1)}\right]^\epsilon dt\right\}.
\end{aligned}
\tag{36}
$$

4.5. Quantile Function

Let $Q_{v,\kappa}(u)$ be the beta quantile function with parameters v and κ. The quantile function of the BKBX model with $\theta = (v, \kappa, \phi, \psi, \vartheta, \tau)$ parameters, let $t = Q(u)$, using the cumulative distribution function given as:

$$
FBKBX(t;\theta) = I1-\{(1-e-(\tau t)2)\vartheta\phi\}\psi(v,\kappa).
$$
$$
\text{Let}, u = I1-\{(1-e-(\tau t)2)\vartheta\phi\}\psi(v,\kappa)
$$

$$I_u^{-1}(\nu, \kappa) = 1 - \{(1 - e^{-(\tau t)^2})^{\vartheta \varphi}\}^{\psi}, \tag{37}$$

multiply both powers by "$1/\psi$" to deduce the equation:

,

$$(38) \quad \left(I_u^{-1}(\nu, \kappa)\right)^{1/\psi} = 1 - \left\{\left\{(1 - e^{-(\tau t)^2})^{\vartheta \varphi}\right\}^{\psi}\right\}^{1/\psi}$$

$$\left(I_u^{-1}(\nu, \kappa)\right)^{1/\psi} = 1 - \left\{(1 - e^{-(\tau t)^2})^{\vartheta \varphi}\right\}.$$

Similarly, we again multiply both powers by "$1/\vartheta\phi$" to deduce the equation:

,

$$(39) \quad \left[1 - \left(I_u^{-1}(\nu, \kappa)\right)^{1/\psi}\right]^{1/\vartheta\varphi} = \left[(1 - e^{-(\tau t)^2})^{\vartheta\varphi}\right]^{1/\vartheta\varphi}$$

$$\left[1 - \left(1 - I_u^{-1}(\nu, \kappa)\right)^{1/\psi}\right]^{1/\vartheta\varphi} = -e^{-(\tau t)^2}.$$

Take a logarithm and multiply for both sides by -1 we obtained:

$$\sqrt{\left\{-\ln\left[1 - \left(1 - I_u^{-1}(\nu, \kappa)\right)^{1/\psi}\right]^{1/\vartheta\varphi}\right\}} = \sqrt{(\tau t)^2}, \tag{40}$$

$$\left\{-\ln\left[1 - \left(1 - I_u^{-1}(\nu, \kappa)\right)^{1/\psi}\right]^{1/\vartheta\varphi}\right\}^{1/2} = \left[(\tau t)^2\right]^{1/2}$$

$$\left\{-\ln\left[1 - \left(1 - I_u^{-1}(\nu, \kappa)\right)^{1/\psi}\right]^{1/\vartheta\varphi}\right\}^{1/2} = \tau t.$$

by taking the square root of both sides:

,

$$(41)$$

Therefore, the quantile function of BKBX distribution is given below by taking the inverse making "t" the subject of the formula.

$$t = \frac{1}{\tau} \times \left\{-\ln\left[1 - \left(1 - I_u^{-1}(\nu, \kappa)\right)^{1/\psi}\right]^{1/\vartheta\varphi}\right\}^{1/2},$$

where, $t = Q_{(u)}$,

$$Q_{(u)} = \frac{1}{\tau} \times \left\{-\ln\left[1 - \left(Q_{\nu,\kappa}(u)\right)^{1/\psi}\right]^{1/\vartheta\varphi}\right\}^{1/2}. \tag{42}$$

Where, $Q_{\nu,\kappa}(u) = 1 - I_u^{-1}(\nu, \kappa)$.

4.6. Skewness and Kurtosis

Skewness: is a measure of symmetry, or more precisely, the lack of symmetry. A distribution, or data set, is symmetric if it looks the same to the left and right of the center point. The **Bowley skewness** by [25], is actually a statistical procedure to find the positive or negative skewed distribution on based on different kinds of data depending on area and the suitability of the functions. It is the most popular tool procedure in finding a skewness fit given by.

$$SK = \frac{Q_{(3/4)} + Q_{(1/4)} - 2Q_{(1/2)}}{Q_{(3/4)} - Q_{(1/4)}},$$

$$SK = \frac{Q_{(0.75)} + Q_{(0.25)} - 2Q_{(0.5)}}{Q_{(0.75)} - Q_{(0.25)}}. \tag{43}$$
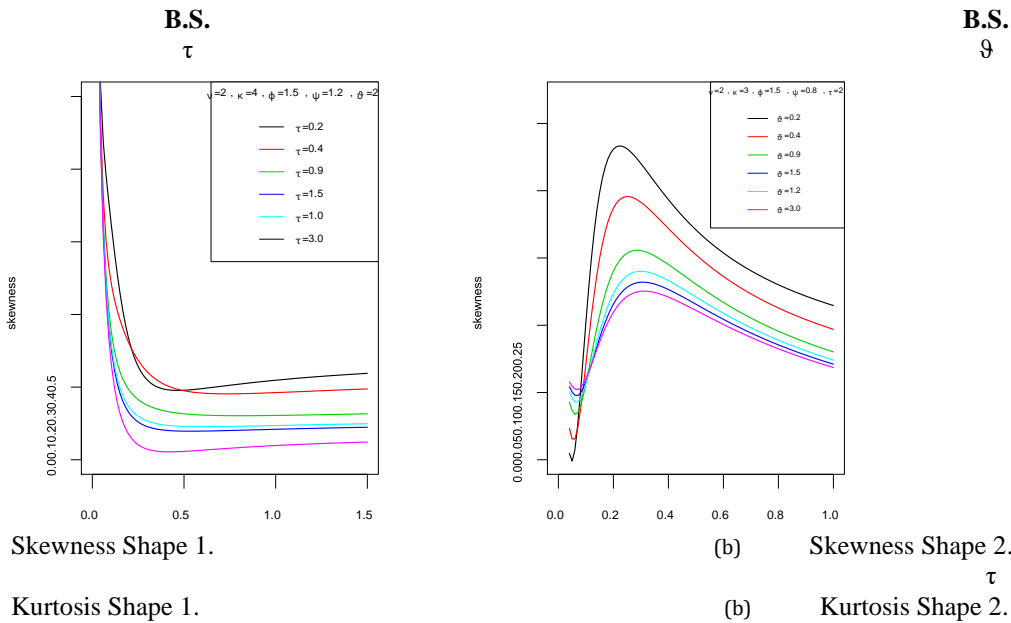
Figure 8: Plot of the BKBX Skewness for
different parameter values of $v,\kappa,\phi,\psi,\vartheta$ and τ.

M.k.   **M.K.**
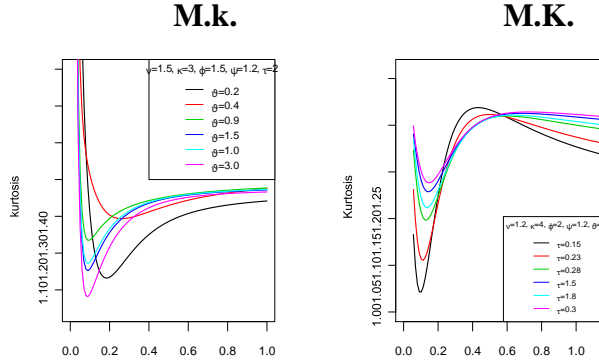
On the other hand we have, **Kurtosis:** is a measure of whether the data are heavy-tailed or light-tailed relative to a normal distribution. That is, data sets with high kurtosis tend to have heavy tails, or outliers. Data sets with low kurtosis tend to have light tails, or lack of outliers. A uniform distribution would be the extreme case. The **Moors Kurtosis** by [26], can be calculated by using the formula given below:

$$MK = \frac{Q_{(1/8)} + Q_{(3/8)} + Q_{(7/8)} - Q_{(5/8)}}{Q_{(3/4)} - Q_{(1/4)}},$$

$$MK = \frac{Q_{(0.125)} + Q_{(0.375)} + Q_{(0.875)} - Q_{(0.625)}}{Q_{(0.75)} - Q_{(0.25)}}. \qquad (44)$$

B.S.   **B.S.**
τ ϑ

(a)   Skewness Shape 1.   (b)   Skewness Shape 2.
ϑ τ
(a)   Kurtosis Shape 1.   (b)   Kurtosis Shape 2.

Figure 9: Plot of the BKBX Kurtosis for different parameter values of $v,\kappa,\phi,\psi,\vartheta$ and τ.

Figure (8), shows measure of relative symmetry as normally zero indicates symmetry. The larger its absolute value the more asymmetric the distribution. The positive parameters values indicates a long right tail skeweness.

Figure (9), shows the measure of the relative peakedness of a distribution. For a normal bellshaped distribution (mesokurtic) and a platykurtic distribution (flatter than a normal distribution with shorter tails). Also a leptokurtic distribution (more peaked than a normal distribution with longer tails) respectively.

4.7. Inference of Estimation:

The most widely used method for the estimation of parameters of distribution is the maximum likelihood estimation method (MLE) and the moment method. We employ MLE to estimate the unknown parameter of BKBX distribution.The BKBX model with six parameters: $\theta = (v, \kappa, \phi, \psi, \vartheta, \tau)$, given the likelihood function as:

$$
\begin{aligned}
L(\theta) = \prod_{i=0}^{n} \Big\{ & \frac{2}{B(\nu, \kappa)} \varphi\psi\vartheta\tau^2 t e^{-(\tau t)^2} \{1 - e^{-(\tau t)^2}\}^{(\vartheta\varphi-1)} \\
& \times [1 - \{1 - e^{-(\tau t)^2)^{\vartheta\varphi}}\}]^{\psi\kappa-1} \\
& \times [1 - [1 - \{(1 - e^{-(\tau t)^2})^{\vartheta\varphi}\}]^{\psi}]^{\nu-1} \Big\}.
\end{aligned}
\tag{45}
$$

for $v > 0, \kappa > 0, \phi > 0, \psi > 0, \vartheta > 0, \tau > 0$. Let $t = (t_1, t_2, .., t_n)^T$ be a random sample of size "n" from BKBX with parameters: $\theta = (v, \kappa, \phi, \psi, \vartheta, \tau)^T$, for i=1,2,...,n. Then the log-likelihood function for is given by:

$$
\begin{aligned}
\ell(\theta) = & n \ln\Big\{ \frac{2\varphi\,\psi\,\vartheta}{B(\nu,\ \kappa)} \Big\} + 2\sum_{i=1}^{n} \ln\left(\tau t_i\right) - \sum_{i=1}^{n}\left(\tau t_i\right)^2 \\
& + \left(\vartheta\varphi - 1\right) \sum_{i=1}^{n} \ln\left[1 - e^{-(\tau t_i)^2}\right] \\
& + \left(\psi\kappa - 1\right) \sum_{i=1}^{n} \ln\left[1 - \left(1 - e^{-(\tau t_i)^2}\right)^{\vartheta\varphi}\right] \\
& + \left(\nu - 1\right) \sum_{i=1}^{n} \ln\left[1 - \left[1 - \left(1 - e^{-(\tau t)^2}\right)^{\vartheta\varphi}\right]^{\psi}\right].
\end{aligned}
\tag{46}
$$

Equation (47), reduces to as follows:

$$
\begin{aligned}
\ell(\theta) = & n\Big(\ln(2) + \ln(\varphi) + \ln(\psi) + \ln(\vartheta) \Big) + 2n\ln(\tau) \\
& + \sum_{i=1}^{n} \ln(t_i) - \sum_{i=1}^{n}(D_i)^2 - n\ln\Big[B(\nu,\kappa)\Big] + \Big(\vartheta\varphi - 1\Big)\sum_{i=1}^{n}\ln\Big(M_i\Big) \\
& + \Big(\psi\kappa - 1\Big)\sum_{i=1}^{n}\ln\Big(R_i\Big) + \Big(\nu - 1\Big)\sum_{i=1}^{n}\ln\Big\{1 - (R_i)^{\psi}\Big\}.
\end{aligned}
\tag{47}
$$

By applying the partial derivatives of the equation (47), that is, the log-likelihood of the BKBX new model above with respect to $\theta = (v, \kappa, \phi, \psi, \vartheta, \tau)$ components of the score vectors: $U_\Phi = (U_v, U_\kappa, U_\phi, U_\psi, U_\vartheta, U_\tau)$ and by denoting:

$$
\begin{aligned}
D_i &= \tau t_i \\
K_i &= e^{-(D_i)^2} \\
M_i &= 1 - e^{-(K_i)^2} \\
R_i &= 1 - \Big(M_i\Big)^{\vartheta\varphi}.
\end{aligned}
$$

were obtained as follows:

$$
\tag{48}
$$

(50)
$$\frac{\partial l}{\partial \nu} = U_\nu = -n\Psi(\nu) + n\Psi(\nu + \kappa) + \sum_{i=1}^{n} \ln[1 - (R_i)^\psi] = 0, \qquad (49)$$

(52)
(53)
$$\frac{\partial l}{\partial \kappa} = U_\kappa = -n\Psi(\kappa) + n\Psi(\nu + \kappa) + \psi \sum_{i=1}^{n} \ln(R_i \qquad (51)$$

$$\frac{\partial l}{\partial \varphi} = U_\varphi = \frac{n}{\varphi} + \vartheta \sum_{i=1}^{n} \ln(M_i) + (1 - \psi\kappa) \sum_{i=1}^{n} \frac{(M_i)^{\vartheta\varphi} \ln M_i}{R_i}$$
$$+ (\nu - 1) \sum_{i=1}^{n} \frac{[\psi[R_i]^{\psi-1}(M_i)^{\vartheta\varphi} \ln(M_i)]}{1 - (R_i)^\psi} = 0.$$

$$\frac{\partial l}{\partial \psi} = U_\psi = \frac{n}{\psi} + \kappa \sum_{i=1}^{n} \ln R_i + (1 - \nu) \sum_{i=1}^{n} \frac{(R_i)^\psi \ln R_i}{1 - (R_i)^\psi} = 0$$

$$\frac{\partial l}{\partial \vartheta} = U_\vartheta = \frac{n}{\vartheta} + \varphi \sum_{i=1}^{n} \ln(M_i) + (1 - \psi\kappa) \sum_{i=1}^{n} \frac{(M_i)^{\vartheta\varphi} \ln M_i}{R_i}$$
$$+ (\nu - 1) \sum_{i=1}^{n} \frac{[\psi[R_i]^{\psi-1}(M_i)^{\vartheta\varphi} \ln(M_i)]}{1 - (R_i)^\psi} = 0.$$

$$\frac{\partial l}{\partial \tau} = \frac{2n}{\tau} + \frac{2}{\tau} \sum_{i=1}^{n} (K_i)^2 + (\vartheta\varphi - 1) \sum_{i=1}^{n} \frac{\tau^2 M_i K_i^2}{M_i}$$
$$+ \left(1 - \psi\kappa\right) \sum_{i=1}^{n} \frac{2\varphi\psi(1 - R_i)^\psi M_i \left(R_i\right)^{\vartheta\varphi-1}}{\left(R_i\right)^\psi}$$
$$+ 2\varphi\vartheta\tau\left(\nu - 1\right) \sum_{i=1}^{n} \frac{(D_i)^2 M_i \left(R_i\right)^{\vartheta\varphi-1}}{1 - \left[(R_i)\right]^\psi} = 0.$$

Where $\omega(.)$ represents the digamma function obtaining the estimations of the score functions in equations: (48), (49), (50), (51), (52) and (53), using the Maple software to simplify the solution iteratively. The provision of interval estimation for the hypothesis test based on the new model unknown parameters, an information matrix of all the BKBX distribution where % < 0, and also the 6 × 6 matrix with $I_n = I_n(\theta)(\theta = (\nu, \kappa, \phi, \psi, \vartheta, \tau))$ is written as follows:

$$I_n = \begin{bmatrix} I\nu\nu & I\nu\kappa & I\nu\phi & I\nu\psi & I\nu\vartheta & I\nu\tau \\ I\kappa\nu & I\kappa\kappa & I\kappa\phi & I\kappa\psi & I\kappa\vartheta & I\kappa\tau \\ & I\phi\kappa & I\phi\phi & I\phi\psi & I\phi\vartheta & \\ & I\psi\kappa & I\psi\phi & I\psi\psi & I\psi\vartheta & I\phi\tau \\ I\phi\nu & I\vartheta\kappa & I\vartheta\phi & I\vartheta\psi & I\vartheta\vartheta & \\ & I\tau\kappa & I\tau\phi & I\tau\psi & I\tau\vartheta & I\psi\tau \\ I\psi\nu & & & & & \\ & & & & & I\vartheta\tau \\ I\vartheta\nu & & & & & I\tau\tau \\ I\tau\nu & & & & & \end{bmatrix}$$

5. Simulation Studies

We conduct a Monte Carlo simulation using the inverse CDF method considering the quantile function of the new BKBX model from Equation (43), used to generate the random numbers for the simulation results and also compare with any existing model for validating the efficiency of developing the new model.

5.1. Monte Carlo Inverse CDF method

In this part we employ an algorithm that generate a sample of size "n" randomly from the *ET−E(ET−E(θ)* model for the parameters ($\theta = v, \kappa, \phi, \psi, \vartheta, \tau$) and the sample size "n". The simulation process has the following steps below:

1. Step 1. We set the sample size "n" and θ.
2. Step 2. We set the initial value t_o for a random procedure.
3. Step 3. We set $j = 1$.
4. Step 4. We generate $U \sim Uniform(0,1)$.
5. Step 5. We set an Update t_o by using the Monte Carlo simulation process.
6. Step 6. If $—t_o − t| \leq \epsilon$, (moderately small, > 0 tolerance limit). Then, $t*$ to the desired sample from $F(t)$.
7. Step 7. If $— t_o − t| >$, then, set $t_o = t*$ and then proceed to step 5.
8. Step 8. We repeat the procedure steps from 4-7, for $j = 1,2,...,n$ and obtained $t_1, t_2,...,tn$.

A simulation studies was done to access and validate the efficiency of the new model as we adopt the Monte Carlo simulation using the inverse CDF method of sampling considering the quantile function above of the new model which we study to access the performance of the MLE's of $\theta = (v, \kappa, \phi, \psi, \vartheta, \tau)$ parameters.

We also generate different "n" sample observation from Equation (43). The parameter estimation was done by maximum likelihood method. We considered different sample size n =(50,150,300,500) and the number of repetition is 1000 respectively. The true parameters value at different sets for each model were used respectively.

Table (1) and Table (2), shows that the mean, bias and MSE of the estimate parameters. We observed that, when we increase sample size "n", the mean is converging towards the true-value while, the bias and MSE are becoming very smaller close to zero for both BKBX and BWB models. Likewise, the blue colour estimated values indicates how the bias and MSE are decreasing with increase of the sample sizes n=(50,150,300,500) respectively.

Table 1: The mean, bias and MSE on Monte Carlo simulation for BKBX parameters values $\theta=(2,2,0.6,0.4,2,2)$ and BWB parameter values $\theta=(2,2,0.4,2)$.

Parameter	True value	Sample size "n"	Mean	Bias	MSE	(AIC)
BKBX						
ν	2	50	2.1822	0.1822	0.0453	
		150	2.1532	0.1532	0.0284	
		300	2.1336	0.1336	0.0200	
		500	2.1247	0.1247	0.0164	
κ	2	50	2.6415	0.6415	0.4242	
		150	2.6283	0.6283	0.4003	
		300	2.6263	0.6263	0.3949	
		500	2.6251	0.6251	0.3917	
φ	0.6	50	0.6172	0.0172	0.3500	
		150	0.6134	0.0134	0.3487	
		300	0.6097	0.0097	0.3401	
		500	0.6085	0.0085	0.2909	(106)
ψ	0.4	50	0.5261	0.1261	0.0473	
		150	0.5066	0.1066	0.0344	
		300	0.4784	0.0784	0.0303	
		500	0.4560	0.0560	0.0223	
ϑ	2	50	2.2172	0.2172	0.0473	
		150	2.1576	0.1576	0.0282	
		300	2.1205	0.1205	0.0223	
		500	2.1071	0.1071	0.0131	
τ	2	50	2.7078	0.7078	0.0319	
		150	2.4729	0.4729	0.0282	
		300	2.2058	0.2058	0.0207	
		500	2.1046	0.1046	0.0135	
BWB						
ν	2	50	2.0286	0.0286	0.2886	
		150	2.0538	0.0538	0.1314	
		300	2.0436	0.0436	0.0716	
		500	2.0394	0.0394	0.0440	
κ	2	50	2.0233	0.0233	0.1840	
		150	2.0317	0.0317	0.1027	
		300	2.0094	0.0094	0.0435	
		500	2.0139	0.0139	0.0284	(112)
φ	0.4	50	0.4205	0.0205	0.0058	
		150	0.4023	0.0023	0.0021	
		300	0.3995	-0.0005	0.0011	
		500	0.3986	-0.0014	0.0007	
τ	2	50	2.1194	0.1194	0.2284	
		150	2.0785	0.0785	0.0930	
		300	2.0785	0.0785	0.0846	
		500	2.0496	0.0496	0.0574	

Table 2: The Bias and RMSE on Monte Carlo simulation for parameters values θ=4,4,1,1,4,4
BWB parameter values θ

Parameter	True value	Sample size "n"	Mean	Bias	MSE(AIC)
BKBX					
		50	4.6421	0.6421	0.4448
		150	4.5841	0.5841	0.3621
ν	4	300	4.5497	0.5497	0.3166
		500	4.5337	0.5337	0.2934
		50	4.7252	0.7252	0.6222
		150	4.6681	0.6681	0.5273
κ	4	300	4.6263	0.6263	0.3949
		500	4.4889	0.4889	0.3446
		50	1.0863	0.0863	0.1037
		150	1.0578	0.0578	0.0935
φ	1	300	1.0397	0.0397	0.0361
		500	1.0173	0.0173	0.0205 (104)
		50	1.0796	0.0796	0.0252
		150	1.0660	0.0660	0.0228
ψ	1	300	1.0814	0.0814	0.0211
		500	1.0523	0.0523	0.0163
		50	4.1024	0.1024	0.5346
		150	4.0843	0.0843	0.4525
ϑ	4	300	4.0615	0.0615	0.4173
		500	4.0342	0.0342	0.3101
		50	4.4565	0.4565	0.3836
		150	4.4251	0.4251	0.3576
τ	4	300	4.4102	0.1302	0.3303
		500	4.4039	0.4039	0.2951

BWB

		50	4.0535	0.0535	0.3878	
		150	4.1326	0.1326	0.2054	
ν	4	300	4.0986	0.0986	0.0959	
		500	4.1073	0.1073	0.0977	
		50	4.1419	0.1419	0.1551	
		150	4.1340	0.1340	0.1050	
κ	4	300	4.1073	0.1073	0.0511	
		500	4.0910	0.0910	0.0498	(109)
		50	1.0352	0.0352	0.0239	
		150	0.9954	-0.0046	0.0071	
φ	1	300	0.9931	-0.0069	0.0039	
		500	0.9908	-0.0092	0.0025	
		50	4.0047	0.0047	0.0678	
		150	3.9990	-0.0010	0.2140	
τ	4	300	3.9967	-0.0033	0.0164	
		500	3.9839	-0.0161	0.0280	

6.   Application of Real Datasets:

Here, we apply data set to clarify the fitness of Beta Kumaraswamy-Burr Type X distribution is a better model than the baselines, Burr type X and Kumaraswamy-Burr Type X distributions respectively.

6.1.   Life of Fatigue Fracture of Kevlar Dataset:

"This data set consists of 306 observations of the life of fatigue fracture of Kevlar dataset, originally obtained by workers at the UK National Physical Laboratory. On the other side they fail to include the units of measurements in the work". The data have also been analyzed by [27].

6.2.   AIDS Clinical Trials Group Study 320 Dataset:

"The data come from a double-blind, placebo-controlled trial that compared the three-drug regimen of indinavir (IDV), open label zidovudine (ZDV) or stavudine (d4T) and lamivudine (3TC) with the two-drug regimen of zidovudine or stavudine and lamivudine in HIV-infected patients" by [28].

6.3.   Nicotine Measurements Dataset:

"The following data is about 346 nicotine measurements made from several brands of cigarettes in 1998. The data have been collected by the Federal Trade Commission which is an independent agency of the US government, whose main mission is the promotion of consumer protection by [29].

6.4.   Criterion:

The criterion like: log-likelihood, Akaike information criterion, consistent Akaike information criterion and Bayesian information criterion for the data set above so as to compare the models and to check which have least or smaller -2`, A_kIC , A_kICC, B_{ayes}IC, K-SM and $p-Value$ values. The distribution of the datasets was skewed to the left, right and symmetric respectively.

Table 3: The ML estimates, -2` log-likelihood, A_kIC , A_kICC and B_{ayes}IC, K-SM and $p - Value$ values. for the life of fatigue fracture of Kevlar dataset.

Model	MLE	-2`(θ) B_{ayes} IC	A_k IC	A_k ICC		K-SM(CvM) Test	Probability (p-value) Cramer-von Mises Watson (W) Test
Beta Kum-Burr Type X	ν =1.3782 κ = 2.4670 ϑ =1.9045 τ= 3.1353 φ =6.349 ψ= 0.905	121	248	256	253 0.2776		0.7841 0.1574 0.1494
Kum-Burr Type X	φ =1.2519 ψ= 0.886 ϑ= 4.646 τ =3.2238	123	252	259	256 0.3826	0.4324	0.2216 0.2513
Burr Type X	ϑ=3.532 τ =1.0269	127	255	266	258 0.3772	0.4018	0.5432 0.5285

Table 4: The ML estimates, log-likelihood, A_k IC, A_k ICC, B_{ayes} IC for the measurements of the AIDS clinical trials group study 320 dataset.

Model	MLE	-2`(θ)	A_k IC	A_k ICC	B_{ayes} IC	K-SM		Probability (p-value) Cramer-von Mises (CvM) Test Watson (W) Test
Beta Kum-Burr Type X	ν =0.0439 κ= 0.5237	φ =0.2158 ψ= 0.2467	220.6718	449.3437	450.2326 456.	ϑ =0.0345 τ= 0.0882		0.8504 0.6214 0.1732 0.184!
Beta-Weibull			φ =0.5237 ψ= 0.0847 ϑ =0.2352 τ= 0.0483	238.1199 482.2399 482.7617 487.9760 0.2542	0.5908		0.184!
Gen. Gompertz			φ =0.0262 ψ= 0.0896 τ =0.0834	222.6718 450.48819 451.0099 476.2242 0.1832	0.2253		0.2698
Gen. Exponential	ϑ=0.0845 231.7653 τ =0.0834	462.8646		464.6096 470.1024	0.1703			0.2145 0.3986 0.3608

7. Summary Of Comments On The Simulation Results

In summary we proposed a new model called Beta Kumaraswamy Burr Type X (BKBX) with six parameters $\theta = (\nu, \kappa, \phi, \psi, \vartheta, \tau)$ that extends and
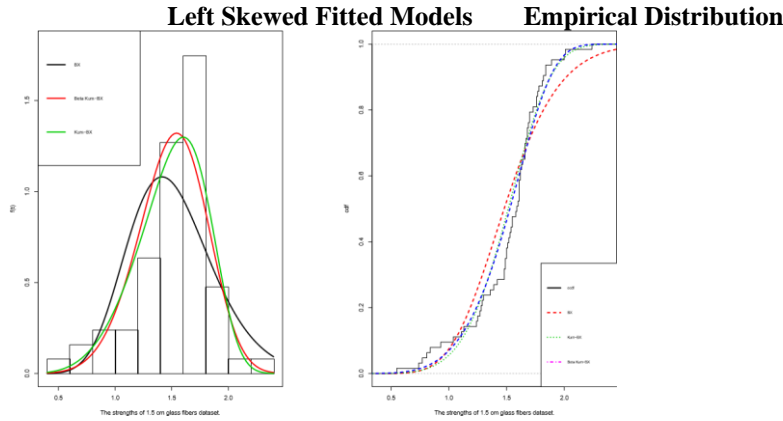
(a) Histogram For Left Skewed   (b) Empirical Distribution 1. Dataset.

Figure 10: Graph of the histogram and empirical distribution for life of fatigue for Kevlar dataset.
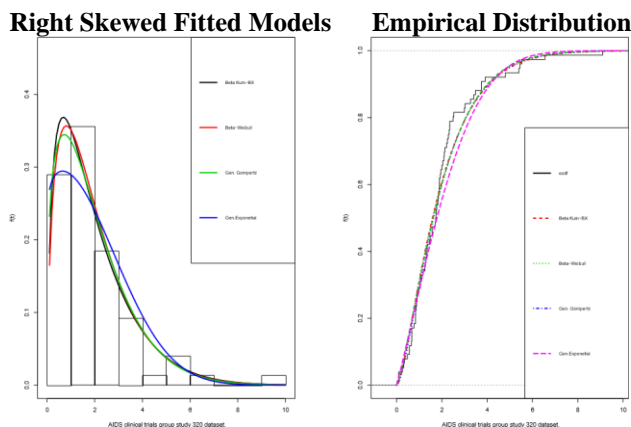
(a)   Histogram   For   Right   (b) Empirical Distribution 2. Skewed Dataset.

Figure 11: Graph of the histogram and empirical distribution for Beta Kum-BX with existing different models of AIDS clinical trials group study 320 dataset.

Table 5: The ML estimates, log-likelihood, A_kIC, A_kICC, B_{ayes}IC for Nicotine measurements dataset.

Model	MLE-2`(θ)	A_kIC	A_kICC	B_{ayes}IC	K-SM	probability (p-value)	Cramer-von Mises (CvM) Test	Watson (W) Test
Beta Kum BX	ν =0.0646 κ= 0.0357 φ= 0.0296 ψ= 0.0324 ϑ =0.0428 τ= 0.0475	97.0835	200.1675	201.3892	203.5740	0.5416	0.7245	0.1206	0.1173
Beta-Weibull	φ =0.0253 ψ= 0.0289 ϑ =0.0834 τ= 0.0243	97.0838	202.1670	201.4307	206.7090	0.7212	0.5304	0.2526	0.2417
Gen. Gompertz	φ =0.0137 ψ τ =0.2352	= 0.03399	9.13442	04.26882	05.53192	07.6752	0.5043	0.5432 0.4187 0.3920
Gen.Exponential	ϑ=0.0742 τ =0.0234	102.9185	209.8371	210.4383	212.1080	0.3490	0.2742	0.5205	0.5124

generalized the Kum-G family, Burr type X distributions and many sub-models by the family of BK-G family which was proposed by [32].

We obtained the distributional properties like: probability density function, cumulative distribution function, hazard function and their expansions. Also, the statistical properties like: quantile function, Browley skewness, Moors Kurtosis were also obtained. The parameters were estimated by using maximum likelihood estimation (MLE) methods. A simulation study for the model parameters was conducted by given different true values as we can see the higher you increase the "n" bias the smaller the mean square error becomes at each level of the simulation process.

Based on the simulation results for the comparison using the Akaike information criterion (AIC) as to check the best model in-terms of providing a meaningful and exact true value and also how the mean, bias and root mean square errors were obtained.

From Table (1) and Table (2), it was shown that as the sample sizes at each level of the process the bias and the mean square error (MSE) decreases with increasing of the sample sizes, where it proven that the BKBX is the best compare to Beta-Weibull model in-terms of comparison but on the other hand the AIC value of BKBX gives the smallest value of the context that not always the best but rather based on the true values used in the simulation study respectively.

7.1. Summary Of Comments On The Real Data

Table 3, 4 and 5 above and below shows MLEs for the individual fitted model for the adopted datasets and the estimated (-2`, A_k IC , A_k ICC and B_{ayes} IC, K-SM and $p-Value$), where the results obtained indicates clearly that the Beta Kumaraswamy-Burr Type X is a stronger with highest peak, flexible and vital to its sub-models and other existing models like: Beta- Weibull, generalized Gompertz and generalized exponential were used here for fitting the data set respectively.

These model as fitted based on the above datasets proves that it is a good example for modeling left, right skewed and symmetric datasets. Also, the

Symmetry Data Fitted Models   Empirical Distribution
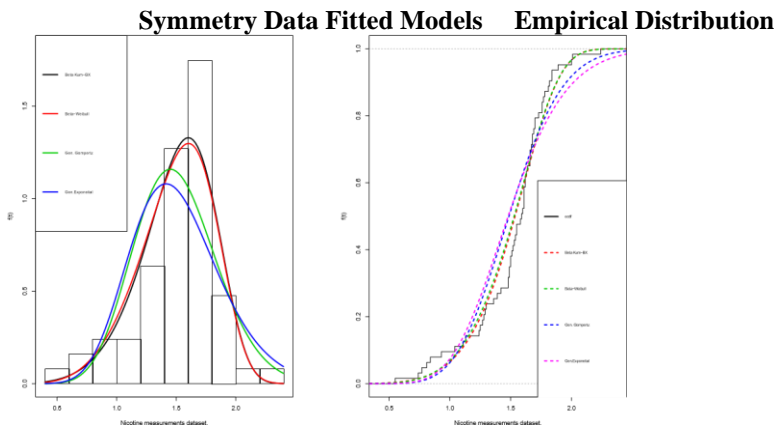
(a) Histogram For Symmetry   (b) Empirical Distribution 3. Dataset.

Figure 12: Graph of the histogram and empirical distribution for Beta Kum-BX with existing different models for the Nicotine Measurements dataset.

likelihood ratio test with the hypothesis $H_o : a = b = 1$ versus $H_1 : a$ 6= 1,b 6= 1. Based on the data above $\omega = 20.567 > 5.991 = \chi_{4;0.05}$, we therefore reject the null hypothesis. Figure 10, 11 and 12 above and below for the plot of densities which compares the models with the histogram empirical graphs proves that Beta Kum-BX is closer to the histogram peak point than the other models which shows the flexibility of the new model with strong properties. On contrast it shows that based on this three applicable datasets from different areas which fits Beta kumaraswamy Burr type X (Beta Kum-BX) distribution having the smallest (-2`, A_k IC , A_k ICC, B_{ayes} IC, K-SM and $p-Value$) among all. This suggest that the Beta Kum-BX distribution is very good in modeling left and right skewed and also some symmetric datasets respectively.

In other to validate the goodness-of-fit for the new model based on the datasets at different level applied like; laboratory dataset, Hiv/Aids (censored) and nicotine measurement for smokers with all comes from science and medical respectively. The used of the test statistics for each model both the new and existing are: Cramer-von Mises (CvM) and the Watson (tests) by [30]. In general view, the suitability of a model based on any datasets apply depends on how smaller the values obtained at each test result for a given datasets where it clearly proves that the Kolmogorov-Smirnov test, $p-Value$, Cramer-von Mises (CvM) and the Watson (tests) were obtained in all the three provided tables showing the flexibility and efficiency of the new models Beta Kum-BX with six parameters having a better fitness as well as the histogram and empirical graphs respectively.Lastly we hope that the proposed model and its generated models will attract wider applications in several areas among others.

8. General Conclusion:

A new model was introduced called Beta Kumaraswamy Burr type X (Beta Kum-BX) with six parameters ($v, \kappa, \vartheta, \psi, \vartheta, \tau$) that extends the Kum-G family, and Burr type X distributions by the family of Beta Kum-G family which was proposed by [8]. We obtain the distributional properties like: PDF, CDF and shapes of hazard function and their expansions. The unknown parameters were estimated using Bayes and MLE methods apply on three (3) different real data sets on medical (normally

fits beta model), hydrological (normally fits Kumaraswamy model) and engineering (normally fits Burr type X model) by using the goodness of fit is clarified.

This new model allocate a better fitness, flexibility and vital to use in medical, survival and reliability analysis area when the data follows and suits a Beta Kumaraswamy Burr type X distribution than some its sub models and also different family models. It is a very good model for symmetric, right and left skewed data sets. We recommend medical practitioners, doctors, engineers, statisticians to adopt this suitable Beta Kum-BX in modeling their large group of data as its consist of three strong models property.

8.1. Acknowledgement

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